Book-Value Wealth Taxation, Capital Income Taxation, and Innovation*

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Abstract

When is a wealth tax preferable to a capital income tax? When is the opposite true? More generally, can capital taxation be structured to improve productivity, incentivize innovation, and ultimately increase welfare? We study these questions theoretically in an infinite-horizon model with entrepreneurs and workers, in which entrepreneurial firms differ in their productivity and are subject to collateral constraints. The stationary equilibrium features heterogeneous returns and misallocation of capital. We show that increasing the wealth tax increases aggregate productivity. The gains result from the "use-it-or-lose-it" effect of wealth taxes when returns are heterogeneous, which causes a reallocation of capital from entrepreneurs with low productivity to those with high productivity. Furthermore, if the capital income tax is adjusted to balance the government's budget, aggregate capital, output, and wages also increase. We then study the welfare maximizing combination of wealth and capital income taxes and show that the optimal mix shifts towards a higher wealth tax and a lower capital income tax as the capital intensity of production increases. For a range of plausible parameter values, the optimal wealth tax is positive, whereas the capital income tax can be positive or negative (a subsidy). We then endogenize the entrepreneurial productivity distribution by introducing either ex ante innovation or entrepreneurial effort in production and show that this strengthens our results: by allowing entrepreneurs to keep more of the upside relative to a capital income tax, a wealth tax incentivizes more innovation and entrepreneurial effort, leading to larger increases in productivity, output, and welfare.

JEL Codes: E21, E22, E62, H21.

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1 Introduction

When is a wealth tax preferable to a capital income tax? When is the opposite true? More generally, can capital taxation be structured to improve productivity, incentivize innovation, and ultimately increase welfare? While these and related questions dominate policy debates, standard economic frameworks are largely silent on them. This is because capital income taxation and wealth taxation are equivalent under the standard assumption that all individuals earn the same rate of return on wealth. However, a growing body of empirical work documents large and persistent heterogeneity in returns across individuals, which challenges this assumption and opens the door for differences in the aggregate and distributional outcomes of these two forms of taxation.[1]

In this paper, we study capital income and wealth taxation when returns are heterogeneous across individuals. We establish conditions under which replacing capital income taxes with wealth taxes generates productivity and welfare gains. We also study the more general problem of the optimal mix of wealth and capital income taxes that maximizes welfare. We then extend the framework by introducing innovation to study how wealth and capital income taxation affect the incentives for innovation, and characterize optimal taxes in this setting.

The framework we employ is fairly standard: an infinite-horizon (perpetual-youth) model with entrepreneurs and workers in which entrepreneurial firms differ in their productivity and are subject to collateral constraints. This is similar to the setup used in many papers reviewed below. Entrepreneurs produce a final good, using a common constant-returns-to-scale technology that combines capital and labor. This good is sold to consumers in a perfectly competitive market. Entrepreneurs have access to a bond market, with zero net supply, through which they can borrow from each other, subject to a collateral constraint. In equilibrium, entrepreneurs with high productivity borrow to invest in their own firms, while those with low productivity lend (at least part of) their wealth. Upon death, entrepreneurs (and workers) are replaced by newborn individuals, who each inherit the same amount of wealth (equal to the average wealth in the economy). Workers supply labor inelastically and are hand-to-mouth, so all the wealth is held by entrepreneurs. All agents have log preferences over consumption.

[1]For empirical evidence on persistent return heterogeneity, see, Campbell, Ramadorai and Ranish (2019), Fagereng, Guiso, Malacrino and Pistaferri (2020), Bach, Calvet and Sodini (2020), and Smith, Zidar and Zwick (2023). For surveys of the literature on capital taxation, see Chari and Kehoe (1999), Golosov, Tsyvinski and Werning (2006), and Stantcheva (2020).

The government taxes wealth and capital income from entrepreneurs to fund its spending in government purchases and transfers to workers. An important feature of the model is that the wealth tax is levied on the *book value* of an entrepreneur's assets and not on the *market value* of the entrepreneurial firm they own, which is a key distinction of the wealth tax we study in this paper, as we discuss in a moment.

The main mechanism that underlies many of our results is that the wealth tax puts the same tax burden on entrepreneurs with the same wealth level *regardless of their productivity*, whereas the capital income tax puts a higher tax burden on more productive entrepreneurs (relative to their wealth). Therefore, the capital income tax effectively punishes more productive entrepreneurs, whereas a wealth tax does not. We call this the "use-it-or-lose-it" effect of the wealth tax. It works by shifting the tax burden from high-productivity to low-productivity entrepreneurs, thereby enabling faster wealth growth for high-productivity entrepreneurs while pruning the wealth of low-productivity entrepreneurs. This effect is absent under capital income taxation.

This mechanism also explains why a wealth tax levied on the book value is more effective than one levied on the market value: two entrepreneurs with the same assets but different productivities have the same book value of wealth but (potentially very) different market values of wealth, as the latter incorporates the productivity (and future returns) of the entrepreneur who operates the firm. Therefore, a market-value wealth tax puts a higher tax burden on more productive entrepreneurs (looking more like a capital income tax), weakening the positive reallocation from the use-it-or-lose-it effect. This is why we propose a *book-value wealth tax* as a more interesting and potentially more effective policy tool than the standard wealth tax based on market values.[2]

We show five main results. First, we establish that there exists a unique stationary equilibrium that exhibits capital misallocation and return heterogeneity when the collateral constraint is "not too loose." We derive the upper bound on the collateral constraint that sustains this equilibrium in terms of model primitives and show that, for a range of plausible parameter values, it allows for borrowing that (far) exceeds the current level of aggregate debt in the US (e.g., measured by the debt-to-GDP ratio). Therefore, this equilibrium does not require unrealistically restrictive collateral constraints. In this heterogeneous-return equilibrium, collateral constraints bind for more productive entrepreneurs, who then earn higher rates of return on wealth than less productive ones.

[2] We also studied the book-value wealth tax quantitatively in Guvenen, Kambourov, Kuruscu, Ocampo and Chen (2023) as we discuss below.

We also show an even stronger result: this is the *only equilibrium possible*—regardless of parameter values—when entrepreneurial productivity is endogenized by introducing costly innovation effort by newborn entrepreneurs (Section 6). Intuitively, this is because the upside potential provided by return heterogeneity is necessary to incentivize entrepreneurs to pay the cost of innovation. To the extent that one believes that high-productivity projects require costly innovation, this result suggests that return heterogeneity is a natural outcome to expect in real life. Another appealing feature of this model is that the stationary wealth distribution has a Pareto right tail, as in the US data, and the thickness of the tail is determined by the rate of return of high-productivity entrepreneurs. These results on the wealth distribution echo those in Jones and Kim (2018).

Second, we show a neutrality result that draws a sharp distinction between the two taxes: the steady-state after-tax returns are independent of the capital income tax but do depend on the wealth tax.[3] In particular, through the use-it-or-lose-it effect, the wealth tax increases the dispersion of after-tax returns, raising the returns of high-productivity entrepreneurs (and lowering the returns of low-productivity entrepreneurs), who in turn own a larger share of aggregate wealth. This reallocation of wealth in response to a wealth tax then increases aggregate productivity.[4] By contrast, capital income taxes do not affect aggregate productivity. These results do not depend on whether the government budget is balanced or not.

When the government balances its budget, there are further gains from wealth taxation. Raising the wealth tax rate allows the government to reduce the capital income tax, which then increases the equilibrium levels of capital, output, and wages. This is because the wealth tax is less distorting than the capital income tax in the presence of return heterogeneity due to its effectiveness in mitigating misallocation as discussed above. The magnitudes of the increase in capital, output, and wages with respect to an increase in the wealth tax depend critically on how much an increase in aggregate productivity translates into higher output, which is determined by the capital intensity of production. This result will be key in characterizing the optimal tax results below.

Third, we study the welfare implications of a marginal increase in the wealth tax (and

[3]This stark result about the independence of equilibrium returns from the capital income tax emerges in our framework due to the combination of log utility and constant returns to scale in production.

[4]Note that all the allocative effects of the wealth tax come from the change in after-tax returns as there is no behavioral response in the present model—saving rates are (endogenously) constant, due to the log utility assumption. In Guvenen et al. (2023), we show that, with a more general CRRA utility, the behavioral savings response *increases* the productivity gains from a wealth tax, which suggests that relaxing the log utility assumption would strengthen the results we establish here.

a reduction in the capital income tax to balance the budget) and provide necessary and sufficient conditions for welfare gains of each type of agent. Workers always benefit from a higher wealth tax—thanks to higher wages—and high-productivity entrepreneurs always benefit because they start life with a higher wealth level and experience faster wealth growth over their lifetime—thanks to higher after-tax returns. Although low-productivity entrepreneurs also start out with higher wealth, they experience slower wealth growth (or faster decline) with a higher wealth tax. The latter effect dominates, leading to welfare losses for this group, unless the capital intensity of production—measured as the output elasticity with respect to capital, α—is unrealistically high. Overall, the average return of entrepreneurs also decreases, leading to welfare losses for entrepreneurs as a whole, again, unless α is very high.

Putting these three pieces together, the aggregate welfare change for the entire population (of workers plus entrepreneurs) from an increase in the wealth tax depends on the magnitudes of the increase in wages and the wealth level (both of which increase with α) versus the loss from the lower average return experienced by entrepreneurs. As a result, the condition for average welfare gain amounts to a lower bound on α, which turns out to be around one-third for a wide range of parameter values.

Fourth, we study the *optimal combination* of capital income and wealth taxes and show that it can be characterized as a function of a lower bound and an upper bound on α. If the capital intensity is above the upper bound, $\alpha > \overline{\alpha}$, the benefits of a reduction in misallocation from the wealth tax (i.e., the rise in wages and wealth) are large enough that the optimal wealth tax is positive and the capital income tax is negative (a subsidy); the signs flip when the capital intensity is below the lower bound, $\alpha < \underline{\alpha} < \overline{\alpha}$, and both taxes are positive in the range between the two thresholds. This interval turns out to be typically quite narrow—between 0.3 and 0.4 for reasonable parameter values.

Fifth, we introduce innovation into our framework and study how it affects optimal capital taxation. We consider innovation along both the extensive and intensive margins, which can be thought of as corresponding to *product innovation* and *process innovation,* respectively (Atkeson and Burstein, 2010). We show that the incentives for innovation along both margins depend on the degree of return dispersion in equilibrium, with higher dispersion increasing the benefit of exerting innovation effort. This introduces an additional channel through which wealth and capital income taxation can affect the economy.

To study the extensive margin of innovation, we assume that entrepreneurs choose how much effort to exert at the outset, which increases the probability of drawing a high-productivity technology (Section 6). In this setting, the wealth tax increases return dispersion, thereby raising the equilibrium level of innovation, and consequently boosting the number of high-productivity entrepreneurs. This additional (extensive margin) benefit of wealth taxation increases the optimal wealth tax level relative to the model with exogenous productivities. By contrast, because the capital income tax has no effect on the return dispersion, it has no effect on innovation and entrepreneurship in our model.

An alternative way to think about entrepreneurial effort is as an ongoing activity, which we model by introducing entrepreneurial effort into the production function and allowing entrepreneurs to make a continuous (intensive margin) choice of effort every period (Section 7). In this setup, the neutrality result we described above no longer holds: the capital income tax dampens the incentives for higher effort by taxing the resulting profits (similar to Jones, 2022). By contrast, the wealth tax does not distort the returns to effort, because the entrepreneur's tax liability is independent of profits, leaving them as residual claimants of the profits generated by their additional effort. As a result, increasing the wealth tax rate and reducing the capital income tax rate further increases output and wages through the incentives for higher entrepreneurial effort.

Finally, in Section 8, we relax the assumption of constant entrepreneurial productivity over the life cycle. We do this in a slightly modified version of the baseline model with infinitely-lived entrepreneurs (no perpetual youth) who are subject to idiosyncratic productivity shocks that follow a first-order Markov process. The persistence of these shocks affects how exposed entrepreneurs are to idiosyncratic variation in their returns, an important mechanism for wealth dynamics highlighted by Atkeson and Irie (2022). We show that all of our theoretical results go through in this setup (with modified formulas that account for the persistence of productivity) as long as entrepreneurial productivity is positively autocorrelated, an empirically well-supported assumption (recall footnote 1).

Related Literature

An important common element in the earlier literature on capital taxation is the assumption of homogenous returns across the population. Because capital income and wealth taxes are equivalent under this assumption, an analysis of the differences between the two taxes is naturally absent from this earlier literature, which focuses on capital income taxation (a short list includes Judd 1985; Chamley 1986; Aiyagari, 1995; Imrohoroglu, 1998; Erosa and Gervais, 2002; Garriga, 2003; Conesa, Kitao and Krueger, 2009; Kitao, 2010; Saez and Stantcheva, 2018; Straub and Werning, 2020).

However, there has been renewed research interest in research on wealth taxation in recent years, partly in response to rising wealth concentration at the top, which led to various policy proposals to tax wealth. Some of these recent papers estimate the behavioral savings response to changes in wealth taxes (Seim, 2017; Jakobsen, Jakobsen, Kleven and Zucman, 2019; Londoño-Vélez and Ávila-Mahecha, 2021; Ring, 2021; Brülhart, Gruber, Krapf and Schmidheiny, 2022), whereas others estimate the migration response of the very rich to a wealth tax (Jakobsen, Kleven, Kolsrud, Landais and Muñoz, 2023; Agrawal, Foremny and Martínez-Toledano, 2023). By contrast, there have been few theoretical studies of wealth taxation, especially when returns are heterogeneous, and to our knowledge, no analysis of the use-it-or-lose it effect of wealth taxes until very recently.[5,6]

In a recent paper, Guvenen, Kambourov, Kuruscu, Ocampo and Chen (2023) build a rich overlapping-generations model with return heterogeneity that matches a wide range of empirical features of the distribution of cross-sectional and lifetime rates of return, the extreme concentration and the Pareto tail of the wealth distribution, and the thicker tail of the capital income distribution, among others. They show quantitatively that there are large efficiency and distributional welfare gains from using wealth taxes instead of capital income taxes. The present paper differs in three important ways. First, we abstract from many of the bells and whistles that were required in that paper for a sound quantitative analysis, and use a simpler and more standard framework. This has two benefits. One, we are able to establish the precise theoretical conditions under which a wealth tax yields efficiency gains, a rise in output, wages, and consumption, as well as how the welfare effects are distributed across the population. Two, these results show that the quantitative conclusions reached in Guvenen et al. (2023) hold more generally—in a standard framework and for a wide range of parameter values—strengthening the conclusions about the advantages of a wealth tax relative to a capital income tax. Second, we characterize the optimal *combination* of capital income and wealth taxes, which was not studied in that paper. Third, and finally, we introduce innovation and entrepreneurial effort, which is absent from that paper, and show that they strengthen the effects of a wealth tax.

The framework we study builds upon a workhorse model (especially, in the firm dynamics, development, and capital misallocation literatures), with heterogeneous firms subject to idiosyncratic productivity shocks and collateral constraints (e.g., Quadrini,

[5]Although Allais (1977) and Piketty (2014) verbally described the use-it-or-lose-it mechanism, they did not study it.

[6]Scheuer and Slemrod (2021) is an excellent survey on wealth taxation that also discusses practical issues in implementation.

2000; Cagetti and De Nardi, 2006; Buera, Kaboski and Shin, 2011; and Boar and Midrigan, 2020). To this, we add the assumption of constant-returns-to-scale in production, following Moll (2014), which affords significant tractability and allows us to establish all of our results analytically. In this sense, our framework is most closely related to Buera and Moll (2015) and Itskhoki and Moll (2019). However, these papers do not study capital income or wealth taxation, which is the main contribution of our paper.

A partial exception is Itskhoki and Moll (2019), who study optimal Ramsey policies and find the optimal policy to impose an upper bound on wages early on in the development process, which boosts profits, particularly for productive entrepreneurs, allowing them to accumulate capital more quickly, in turn mitigating the effect of collateral constraints. This is similar to the effects of a wealth tax in our model, which also boosts the after-tax profits of high-productivity entrepreneurs, relaxing their constraints, in turn raising efficiency and incentivizing innovation.

The framework we use is also closely related to the literature on power law models that can generate a Pareto tail for the wealth (and income) distribution (see, among others, Champernowne, 1953; Jones, 2015; Gabaix et al., 2016; and the review in Benhabib and Bisin, 2018). Especially related is Benhabib, Bisin and Zhu (2011) who consider an overlapping-generations model with return heterogeneity and study how the properties of the Pareto tail depend on the model's parameters, including on the estate tax rate. However, they do not study capital income or wealth taxation and return heterogeneity is modeled as an exogenous process, so they do not analyze the macroeconomic implications of the model, as we do here.

Our results for the distribution of wealth and the role of entrepreneurial effort and innovation are also related to those of Jones and Kim (2018) and Jones (2022), derived in the context of the distribution and taxation of top incomes. Our model shows how wealth taxes affect the tail of the wealth distribution through their effects on entrepreneurial returns and how taxing the book-value of wealth instead of capital income incentivizes effort by reducing the effective tax rate on the income of the most productive entrepreneurs, leading to higher aggregate productivity levels as well as higher wealth concentration.

Two contemporaneous papers also explore the effects of wealth taxation. Boar and Midrigan (2023) study the tradeoff between capital income, wealth taxes, and lump-sum transfers to workers. Their framework differs from ours in two key respects: entrepreneurs are infinitely lived (no perpetual-youth), and production exhibits decreasing-returns-to-scale, so there is an optimal scale. The combination of these two features imply that, in the long-run they analyze, most firms accumulate enough capital for the constraints not to bind, leaving little room for financial frictions to matter (small misallocation). In contrast, in our model, the combination of perpetual-youth and constant-returns-to-scale ensures that misallocation still matters, and the wealth tax is an effective tool for mitigating it.[7] Also related is the work of Gaillard and Wangner (2022) who focus on the role of increasing returns to scale in shaping wealth accumulation, adopting a reduced-form return function that allows returns to be increasing in the level of wealth so as to capture scale effects. We instead use constant returns to scale to endogenously generate heterogeneous returns and analytically characterize the equilibrium effects of changes in wealth and capital income taxes. In this sense, these papers are complementary to ours.

2 Model

We begin with a model in which entrepreneurs' productivity distribution is exogenously given. In Sections 3 to 5, we study the effects of wealth and capital income taxation in this setting. Then, in Sections 6 and 7, we endogenize the productivity distribution by introducing innovation and entrepreneurial effort and study how the two taxes affect incentives for innovation and characterize the optimal wealth and capital income taxes in this setting.

2.1 Model Description

Time is discrete. The economy is populated by overlapping generations of homogenous workers (size L) and heterogeneous entrepreneurs (size 1), with perpetual-youth life cycles: both types of agents face a constant probability of death $1-\delta$ each period, and upon death, they are replaced by a cohort of newborns of appropriate size to keep the population constant over time.

Workers and entrepreneurs share the same preferences, defined over consumption:

$$\sum_{t=1}^{\infty} (\beta\delta)^{t-1} \log(c_t), \tag{1}$$

where β is the time discount factor. Workers supply labor inelastically, receive transfers T from the government, and live hand-to-mouth (and therefore hold no wealth). Entrepreneurs

[7] In Guvenen et al. (2023), we study a lifecycle model with decreasing returns to scale, in which case there is still a large enough fraction of firms that are constrained, and the wealth tax delivers substantial efficiency and welfare gains. So, constant-returns-to-scale is not necessary for misallocation to matter.

are the sole capital/wealth owners in the economy. The total wealth of all entrepreneurs who die in a period is distributed equally to (i.e., inherited by) all newborn entrepreneurs. Because the population is constant and mortality risk is independent of age, wealth, and productivity, this starting wealth equals the average wealth in the economy, which we denote with $\bar{a}$.[8]

Each entrepreneur is born with a fixed (exogenous) idiosyncratic productivity, z, and produces a homogeneous good combining capital, k, and labor, n, using a constant-returns-to-scale technology:[9]

$$y = (zk)^{\alpha} n^{1-\alpha}. \tag{2}$$

Entrepreneurs hire labor at wage rate w and can borrow in a bond market at interest rate r to invest in their firm, over and above their own wealth a. Both markets are perfectly competitive. The same bonds, which are in zero net supply, can be used as a savings device, which will be optimal for entrepreneurs whose return in equilibrium is lower than the interest rate r. Thus, k can be greater or smaller than a. Entrepreneurs' borrowing is subject to a collateral constraint that depends on their beginning-of-period wealth (a), that is,

$$k \leq \lambda a, \tag{3}$$

where $\lambda \geq 1$. When $\lambda = 1$, an entrepreneur can use only his wealth in production.[10]

The Government. The government taxes capital income at rate τ_k and (beginning-of-period) book-value of wealth, a, at rate τ_a to finance exogenous expenditures, G, and transfers to workers, T.

2.2 Entrepreneur's Problem

Entrepreneurial Productivity. Entrepreneurial productivity can take on two values: high, z_h, or low, z_ℓ.[11] Each entrepreneur is born with productivity z_h or z_ℓ, with probability

[8] An alternative assumption would be to assign each newborn to an entrepreneur who dies that period ("parent") and assume that the newborn inherits the wealth of that parent. This case delivers essentially the same results, as we show in Section 8.

[9] For convenience, we assume no depreciation, but this is easy to relax.

[10] This specification of the collateral constraint is analytically tractable and is widely used in the literature (see, for example, Banerjee and Newman, 2003; Buera and Shin, 2013; and Moll, 2014). It can also be motivated as resulting from an underlying limited commitment problem (see Guvenen et al., 2023 for further discussion). The importance of financial constraints has broad empirical support; see, e.g., Gomes et al., 2006; Hvide and Møen, 2010; Duygan-Bump et al., 2015; Benmelech et al., 2019; Ring, 2023.

[11] Allowing more values of z (or even a continuous distribution) is fairly straightforward but comes at the cost of notational complexity without adding new insights, so we do not pursue that approach.

μ and $1-\mu$, respectively, which then equals the population shares of high-productivity entrepreneurs (which we refer to as "H-type") and low-productivity entrepreneurs ("L-type"), respectively. We endogenize the distribution of z in Sections 4 and 5.

Entrepreneurial Production. Entrepreneurs choose k and n every period to maximize profit, $\Pi(z,a)$, taking prices as given:

$$\Pi^{\star}(z,a) = \max_{k\leq\lambda a,\, n\geq 0} \left[(zk)^{\alpha} n^{1-\alpha} - rk - wn\right], \tag{4}$$

which yields their labor demand function:

$$n^{\star}(z,k) = \left(\frac{1-\alpha}{w}\right)^{1/\alpha} zk. \tag{5}$$

Substituting (5) into (4), yields the optimal capital choice:

$$k^{\star}(z,a) = \underset{k\leq\lambda a}{\operatorname{argmax}} \left[\alpha\left(\frac{1-\alpha}{w}\right)^{(1-\alpha)/\alpha} z - r\right] k. \tag{6}$$

The constant-return-to-scale technology implies that entrepreneurs whose marginal return to capital (first term in equation 6) is greater than r borrow up to their collateral constraint, and set $k^{\star} = \lambda a$, whereas those whose marginal return is below r do not produce and instead lend all their wealth in the bond market to earn return r. Therefore, optimal entrepreneurial income can be written as $\Pi^{\star}(z,a) = \pi^{\star}(z) \times a$, where

$$\pi^{\star}(z) \equiv \begin{cases} \left(\alpha\left(\frac{1-\alpha}{w}\right)^{\frac{1-\alpha}{\alpha}} z - r\right)\lambda & \text{if } \alpha\left(\frac{1-\alpha}{w}\right)^{\frac{1-\alpha}{\alpha}} z > r \\ 0 & \text{if } \alpha\left(\frac{1-\alpha}{w}\right)^{\frac{1-\alpha}{\alpha}} z \leq r, \end{cases} \tag{7}$$

is the *excess return* an entrepreneur earns above r.

Entrepreneurial Savings. Entrepreneurs' consumption-savings problem is separable from their production problem. In anticipation of our focus below on the stationary equilibrium of the model, we write the recursive problem as a stationary Bellman equation:

$$V(a,z) = \max_{a',c} \log(c) + \beta\delta V(a',z) \tag{8}$$

$$\text{s.t. } c + a' = R(z)\,a,$$

where

$$R(z) \equiv (1 - \tau_a) + (1 - \tau_k)(r + \pi^\star(z)) \tag{9}$$

is the *after-tax gross return on savings,* and the time-invariant taxes τ_a and τ_k, and prices r and w, are taken as given. Importantly, the wealth tax is levied on the *beginning-of-period* wealth, so only the capital income tax is levied on the income flow generated during the period $(r + \pi^\star(z))$. The optimal savings rule for this problem is

$$a'(a, z) \quad = \quad \beta\delta R(z)\, a, \tag{10}$$

which is linear in wealth, with a net savings rate of $\beta\delta$ that is independent of productivity (thanks to log utility), although the gross savings rate (or the growth rate of their wealth) does depend on z through the rate of return they earn, $R(z)$. Therefore, all the reallocation effects of changes in taxation operate through their effect on returns. The details of these derivations can be found in Appendix A.1.[12]

2.3 Recursive Stationary Competitive Equilibrium

We begin with an informal discussion of the types of stationary equilibria that are possible in this model and the particular equilibrium we will focus on in the rest of the paper. Depending on the values of model parameters, three types of stationary equilibria are possible: *(i)* an equilibrium that features capital misallocation, return heterogeneity ($R_h > R_\ell$), and a non-degenerate wealth distribution; *(ii)* an equilibrium without misallocation because all capital is used by the H-type, with no misallocation or return heterogeneity, and *(iii)* a third equilibrium that cycles between these two. In the next section, we show that as long as the collateral constraint satisfies an upper bound, $\lambda < \bar{\lambda}$, the "heterogeneous-return" equilibrium described in *(i)* emerges as the unique stationary equilibrium of the model. We derive the upper bound $\bar{\lambda}$ in terms of model parameters (Assumption 1) and argue that, for a wide range of plausible parameter values, this bound is easily satisfied, which suggests that this is the most relevant equilibrium to focus on. In addition, with misallocation and return heterogeneity, this is the only equilibrium that provides an interesting setting for analyzing wealth and capital income taxation. *Therefore, we focus on this heterogeneous-return equilibrium in the rest of the paper.*[13]

[12]As noted in footnote 4, the effects of the wealth tax would likely be stronger with a utility function with a savings response.

[13]Later, in Section 6, we show that when innovation effort is endogenized, the heterogeneous-return equilibrium is the only stationary equilibrium possible regardless of parameter values. This would also be the unique equilibrium if the distribution of productivity were continuous. Then, the equilibrium would be

From this point on, we proceed in two steps. We first define some key variables and derive some equations that hold in the heterogeneous-return equilibrium. In particular, we give an aggregation result in Lemma 1 that will be useful in subsequent results. We then give an intuitive discussion of how the bond and labor markets work, before we present the existence and uniqueness of equilibrium in the next section.

Defining some key variables. An important feature of our model is that aggregate variables can be expressed in closed form as functions of aggregate capital

$$K \equiv \mu A_h + (1-\mu) A_\ell, \tag{11}$$

(where A_h and A_ℓ are the aggregate wealth of the H-type and L-type, respectively) and aggregate productivity Z (as in Moll, 2014). The latter is endogenous and equal to the wealth-weighted average of two individual-level productivity terms:

$$Z \equiv s_h z_\lambda + (1-s_h) z_\ell, \tag{12}$$

where

$$s_h \equiv \frac{\mu A_h}{\mu A_h + (1-\mu) A_\ell} \tag{13}$$

is the wealth share of the H-type, and

$$z_\lambda \equiv z_h + (\lambda - 1)(z_h - z_\ell) \tag{14}$$

is the *effective productivity of wealth* of the H-type, that is, the return they earn from their own wealth, captured by z_h, plus the excess return from borrowed capital, $(\lambda-1)(z_h - z_\ell)$. Notice that $z_\lambda > z_h$.

Armed with these definitions, we can now state Lemma 1, which shows that aggregate output can be written as a function of aggregate variables only (aggregation) and gives expressions for all equilibrium prices.

Lemma 1. ***(Aggregate Variables in Equilibrium)*** *In the stationary heterogeneous-return equilibrium defined in Proposition 7, aggregate output, the wage rate, the interest*

characterized by a threshold value of productivity above which entrepreneurs borrow, as in Moll (2014).

rate, and gross returns are given as follows:

$$Y = (ZK)^{\alpha} L^{1-\alpha} \tag{15}$$

$$w = (1-\alpha)(ZK/L)^{\alpha} \tag{16}$$

$$r = \alpha (ZK/L)^{\alpha-1} z_{\ell} \tag{17}$$

$$R_{\ell} = (1-\tau_a) + (1-\tau_k)\,\alpha (ZK/L)^{\alpha-1} z_{\ell} \tag{18}$$

$$R_h = (1-\tau_a) + (1-\tau_k)\,\alpha (ZK/L)^{\alpha-1} z_{\lambda}. \tag{19}$$

The proofs of all lemmas and propositions can be found in Appendix B. The expressions given for r, R_{ℓ},and R_h are intimately related to the structure of the equilibrium in the bond market.

Bond market equilibrium. With two levels of productivity, it is easy to see that the H-type will (weakly) demand funds for production and the L-type will (weakly) supply them for saving. The market-clearing interest rate must be between the marginal return to capital of the two types, that is,

$$\alpha \left(\frac{ZK}{L}\right)^{\alpha-1} z_{\ell} \quad \leq r \leq \quad \alpha \left(\frac{ZK}{L}\right)^{\alpha-1} z_h, \tag{20}$$

which is obtained by substituting the equilibrium wage (16) into (7).

The maximum amount the L-type can lend is given by their total wealth, $(1-\mu) A_{\ell}$, while the collateral constraint ensures that the H-type can borrow at most $(\lambda - 1)\,\mu A_h$. The heterogeneous-return equilibrium corresponds to the case when the L-type have more wealth to lend than what the H-type are able to borrow—that is, there is an excess supply of funds in the economy:

$$(\lambda - 1)\,\mu A_h < (1-\mu) A_{\ell}. \tag{21}$$

Clearly, this happens when the H-type are not "too rich" relative to the L-type or when the collateral constraint is "not too loose," or both. Indeed, the inequality in 21 can be shown to simplify to $s_h < 1/\lambda$, which combines the two conditions mentioned. In this case, the H-type borrow up to the collateral constraint—hence $K_h = \lambda A_h$—while the L-type compete with each other to lend, bidding down the equilibrium interest rate to their marginal product (giving $r = \alpha (ZK/L)^{\alpha-1} z_{\ell}$ in Lemma 1).[14] So, their average capital is

[14]An alternative to the market structure we consider here would be to introduce a corporate sector (e.g.,

given by:

$$K_\ell = \frac{(1-\mu)\,A_\ell - (\lambda-1)\,\mu A_h}{1-\mu} > 0. \tag{22}$$

Two properties of this equilibrium we mentioned earlier follow from this discussion. First, the fact that K_ℓ is positive immediately implies that there is capital misallocation. Second, from equations (18) and (19) in Lemma 1, R_ℓ and R_h depend on z_ℓ and z_λ, respectively, so the equilibrium features return heterogeneity.

Labor market equilibrium. The labor demand function in (5) is linear in capital, so it can be aggregated to express the labor market clearing condition as

$$\mu n^\star\left(z_h, K_h\right) + (1-\mu)\, n^\star\left(z_\ell, K_\ell\right) = L. \tag{23}$$

Substituting in n^* from (5) delivers the expression for the equilibrium wage in Lemma 1.

The definition of a stationary recursive competitive equilibrium is standard and is stated in Appendix A.2.

3 Characterization of Stationary Equilibrium

In this section, we first derive two equations that determine the steady-state levels of K and Z in the heterogeneous-return equilibrium and show how they depend on the wealth and capital income taxes. These results will be important for understanding the main assumption that underlies the equilibrium existence and uniqueness results, which we will prove next. We will then show that the stationary wealth distribution has a Pareto right tail.

3.1 Steady State Levels of K and Z

The assets of the L-type and H-type evolve according to:

$$A_i^{'} = \delta^2 \beta R_i A_i + (1-\delta)\,\overline{a}, \tag{24}$$

with technology $Y_c = (z_c K_c)^\alpha L_c^{1-\alpha})$ that faces no collateral constraints, which would provide entrepreneurs an alternative investment option for their wealth. The marginal return of capital in the corporate sector imposes a lower bound on equilibrium r: $r \geq \alpha z_c \,(ZK/L)^{\alpha-1}$. As long as z_c satisfies $z_\ell < z_c < z_h$, both the corporate sector and the H-type produce in equilibrium, while the L-type lend all their wealth and do not produce. This structure delivers the same result as our benchmark model, with z_c replacing z_ℓ.

where $\overline{a} \equiv (1-\mu) A_\ell + \mu A_h = K$ is the wealth of a newborn entrepreneur, which in turn is equal to the average wealth in the economy. Adding up these equations for $i = \{h, \ell\}$ and substituting $\bar{a} = K$, we obtain the law of motion for aggregate capital:

$$\frac{K'}{K} = \delta^2 \beta \underbrace{(s_h R_h + (1-s_h) R_\ell)}_{\text{wealth-weighted avg. return}} + (1-\delta), \tag{25}$$

which shows that the growth rate of K depends on the wealth-weighted average return. Substituting in the expressions for R_ℓ and R_h from Lemma 1 and setting $K' = K$ yield the following equation—that is analogous to the steady-state condition in the neoclassical growth model—where the (after-tax) marginal product of capital is equal to the inverse of the effective discount factor:

$$(1-\tau_a) + (1-\tau_k) \alpha Z^\alpha \left(\frac{K}{L}\right)^{\alpha-1} = \frac{1}{\beta\delta}. \tag{26}$$

This equation has far-reaching implications and plays a critical role in our subsequent results. In particular, it reveals *a neutrality result* that draws a sharp distinction between the two taxes: the after-tax marginal product of capital is *independent of the capital income tax but does depend on the wealth tax.* Rearranging (26) makes this easier to see:

$$(1-\tau_k) \alpha Z^\alpha \left(\frac{K}{L}\right)^{\alpha-1} = \frac{1}{\beta\delta} - 1 + \tau_a. \tag{27}$$

If we change the wealth tax rate τ_a, this has the same effect as changing the effective discount factor and hence changes the after-tax marginal product on the left hand side, whereas if we change τ_k, this causes K or Z or both to adjust so as to keep the after-tax marginal product constant and equal to $(\frac{1}{\beta\delta} - 1 + \tau_a)$. Because entrepreneurs' rates of return, R_h and R_ℓ depend on the after-tax marginal product (eqs. 19 and 18), τ_a increases the levels and dispersion of returns, whereas the τ_k has no effect. The following proposition formalizes these results.

Proposition 1. ***(Capital Income Tax is Neutral for Returns. Wealth Tax is Not)*** *In the stationary heterogeneous-return equilibrium, the after-tax returns of the H-type and L-type are independent of the capital income tax rate but do depend on the wealth tax rate:*

$$R_\ell = 1 - \tau_a + \left(\frac{1}{\beta\delta} - (1-\tau_a)\right)\frac{z_\ell}{Z} \quad \textit{and} \quad R_h = 1 - \tau_a + \left(\frac{1}{\beta\delta} - (1-\tau_a)\right)\frac{z_\lambda}{Z}. \tag{28}$$

In particular, the wealth tax has a "use-it-or-lose-it" effect that changes the dispersion of returns and therefore the level of wealth inequality, whereas the capital income tax has no distributional effects.

Intuitively, these results reflect the fact that τ_k affects the marginal return of capital for both types proportionally and is therefore neutral from a distributional standpoint, while τ_a affects gross returns additively and therefore has a disproportionate (negative) effect on the returns of the L-type. Notice, however, that the proposition does not specify whether the wealth tax increases or decreases inequality. While it is clear from (28) that the direct, use-it-or-lose-it reallocation (or partial equilibrium) reallocation effect of τ_a is to increase the dispersion of returns and therefore inequality, there is also the indirect (general equilibrium) effect through the change in Z when τ_a changes, which needs to be considered. We turn to this next.

Stationary level of Z Equation (26) provided the first condition for the steady state levels of K and Z. We now impose the second condition that ensures a stationary equilibrium—that the wealth share of each type are constant—by evaluating the law of motion for each type in (24) at $A_i' = A_i$ for $i \in \{h, \ell\}$. This implies

$$A_i = \frac{1-\delta}{1-\delta^2 \beta R_i} \overline{a}. \tag{29}$$

Substituting in the definition of $\overline{a} = (1-\mu) A_\ell + \mu A_h$, we obtain

$$1 = (1-\delta) \frac{1-\delta^2 \beta \left((1-\mu) R_h + \mu R_\ell\right)}{\left(1-\delta^2 \beta R_\ell\right)\left(1-\delta^2 \beta R_h\right)}. \tag{30}$$

We then use the stationary value of returns from (28), which yields a quadratic equation that determines the steady state level of Z:

$$\begin{aligned}\left(1-\delta^2 \beta \left(1-\tau_a\right)\right) Z^2 - \left[(1-\delta)\left(\mu z_\lambda + (1-\mu) z_\ell\right) + \delta \left(1-\delta \beta \left(1-\tau_a\right)\right)\left(z_\lambda + z_\ell\right)\right] Z \\ +\delta \left(1-\delta \beta \left(1-\tau_a\right)\right) z_\ell z_\lambda = 0.\end{aligned} \tag{31}$$

This equation reveals a few key properties of the stationary equilibrium. First, we can show that only the larger root of this quadratic equation satisfies $z_\ell < Z < z_\lambda$ (see the appendix). Therefore, if the stationary heterogenous-return equilibrium exists (as we assumed so far), it is also unique. Plugging this value of Z into (26) then determines the steady state level of K. Second, and more important, while τ_a appears in (31), τ_k does not, which means that the capital income tax rate has no effect on equilibrium productivity level—only τ_a does. This result further sharpens the neutrality results in Proposition (1) by adding Z to the list of variables that τ_k has no effect on.[15]

Elasticity of Capital with Respect to Taxes. Before moving forward, we highlight the implications of equation (26) for the response of aggregate capital to taxes as this will be important for our tax results later. Crucially, the wealth tax affects the level of capital through two channels—directly, by changing the right hand side of (26) as well as indirectly, through its effect on productivity (which in turn changes the capital level)—whereas the capital income tax only has the former effect. This asymmetry implies different elasticities of capital with respect to each tax:

$$\xi^K_{\tau_a} \equiv \frac{d\log K}{d\tau_a} = \frac{\alpha}{1-\alpha}\frac{d\log Z}{d\tau_a} - \frac{1}{(1-\alpha)\left(\frac{1}{\beta\delta} - (1-\tau_a)\right)}; \quad (32)$$

$$\xi^K_{\tau_k} \equiv \frac{d\log K}{d\tau_k} = -\frac{1}{(1-\alpha)(1-\tau_k)}. \quad (33)$$

For standard parameter values these elasticities are well within the range reported in Scheuer and Slemrod (2021) in their review of the literature and are consistent with the values reported by Jakobsen, Jakobsen, Kleven and Zucman (2019) (see Figure F.2 in the Appendix). We will have more to say about these formulas in the tax analysis.

3.2 Existence and Uniqueness of Stationary Equilibrium

As mentioned above, the types of equilibria that emerge depend on the tightness of the collateral constraint. We now formally define an upper bound on the collateral constraint in terms of model parameters that ensures the existence and uniqueness of the heterogenous-return equilibrium.

Assumption 1. *The collateral constraint is "not too loose," that is, λ satisfies the following bound:*

$$\lambda < \overline{\lambda} \equiv 1 + \frac{(1-\delta)(1-\mu)}{(1-\delta)\mu + \delta(1-\delta\beta(1-\tau_a))\left(1-\frac{z_\ell}{z_h}\right)}. \quad (34)$$

[15] We should note that this stark conclusion of *complete* neutrality follows from the combination of the constant-returns-to-scale and log utility assumptions. That said, this result still suggests that even in a more general model that relaxes these assumptions, the wealth tax is likely to have a stronger impact on distributional outcomes as well as on productivity than does the capital income tax. Our quantitative results in Guvenen et al. (2023) confirms this conjecture.

Two comments are in order. First, in addition to model parameters, $\bar{\lambda}$ also depends on—and, in particular, is decreasing in—the wealth tax rate. The reason is that a higher τ_a increases wealth inequality (Proposition 1), thereby shifting wealth from the L-type to the H-type ($s_h \uparrow$), which makes it harder for the excess supply condition, $(\lambda - 1)\,\mu A_h < (1-\mu)\,A_\ell$, to hold, unless λ is reduced. Therefore, $\bar{\lambda}$ must get tighter to disallow high values of λ.[16] Second, τ_k does not appear in (34), which is a direct consequence of the neutrality result.

Assumption 1 can be equivalently stated as an upper bound on τ_a for a given λ, by inverting (34). This alternative form provides another useful way to think about the constraint imposed on the economy—as an upper bound on the policy instrument τ_a that sustains this equilibrium for a given value of the primitive λ. Assumption 2 gives this alternative formulation.

Assumption 2 (1'). *Assumption 1 can be stated equivalently as an upper bound on the wealth tax, given λ:*

$$\lambda < \bar{\lambda} \Leftrightarrow \quad \tau_a < \bar{\tau}_a \equiv 1 - \frac{1}{\beta\delta}\left(1 - \frac{1-\delta}{\delta}\frac{1-\lambda\mu}{(\lambda-1)\left(1-\frac{z_\ell}{z_h}\right)}\right). \tag{35}$$

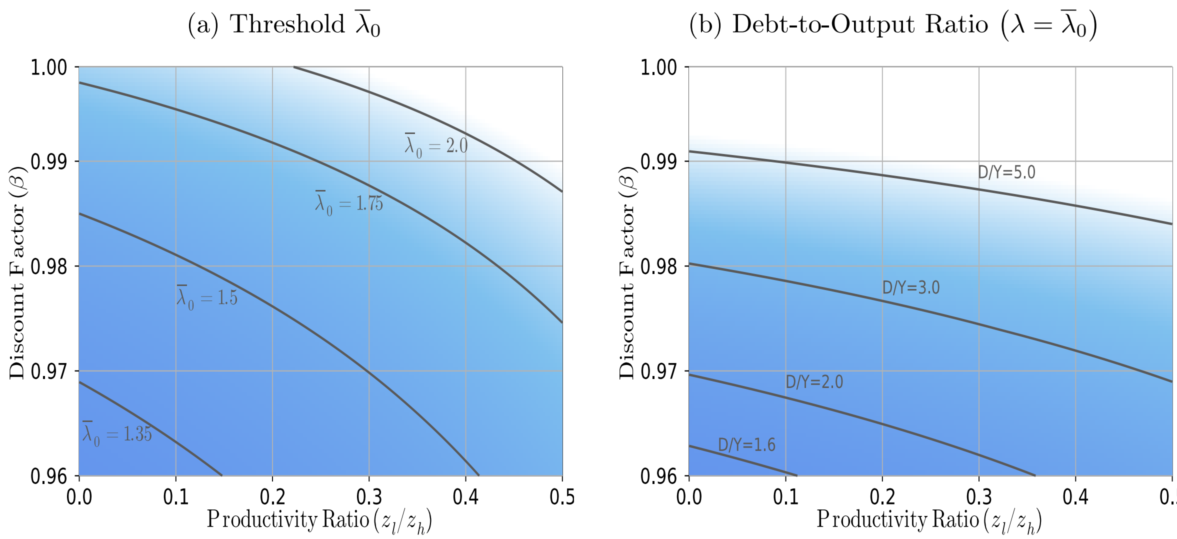
We are now ready to state the next proposition.

Proposition 2. ***(Existence and Uniqueness of Stationary "Heterogeneous-Return" Equilibrium)*** *A stationary competitive equilibrium exists and is unique if and only if λ satisfies Assumption 1. This equilibrium is characterized by an endogenous productivity level Z that satisfies $z_\ell < Z < z_h$, and features return heterogeneity ($R_h > R_\ell$). In addition, the wealth share of the H-type satisfies $s_h < 1/\lambda$.*

So, how much borrowing does Assumption 1 allow for empirically reasonable parameter values? To get a sense about this, in Figure 1, we plot the values of $\bar{\lambda}$ (left panel) and the debt-to-GDP ratio (right panel) for different values of β and z_ℓ/z_h and setting $\tau_a = 0$.

[16] It is also easy to see from (34) that $\bar{\lambda}$ is decreasing in μ and z_h/z_ℓ because, again, higher values of both make it easier $s_h < 1/\lambda$ to be violated. $\bar{\lambda}$ is decreasing in δ because longer life-spans benefit the accumulation of assets by H-type entrepreneurs, who have higher gross returns and hence a higher saving rate out of wealth than the L-types. By contrast, $\bar{\lambda}$ is increasing in β (for a fixed δ) because as both types become more patient the aggregate savings in the economy increase, in turn lowering wealth inequality due to the redistribution of wealth among newborns. This results in a higher wealth share of the L-types, increasing the supply of funds available and reducing the demand (reducing λA_h), sustaining the excess supply equilibrium.

Figure 1: Conditions for Stationary Equilibrium with Heterogeneous Returns



Note: The left panel plots the value of $\overline{\lambda}_0$ found in Proposition 2 for combinations of β and z_ℓ/z_h. The right panel shows the debt-to-output ratio when $\lambda = \overline{\lambda}_0$ computed as $\left(\overline{\lambda}_0 - 1\right) A_h/Y$. In both panels, the remaining parameters are set as follows: $\delta = 49/50$, $\mu = 0.10$, $z_h = 1$, $\tau_k = 25\%$, and $\alpha = 0.4$.

This threshold, which we denote with $\bar{\lambda}_0 \equiv \bar{\lambda}|_{\tau_a=0}$, provides an upper bound for $\bar{\lambda}$ that characterizes the maximum collateral constraint that sustains the equilibrium in the model without a wealth tax.[17] As seen in the left panel of Figure 1, for a plausible value of $\beta = 0.97$ and $z_\ell/z_h = 0.5$, the implied $\overline{\lambda}_0$ is above 1.5, which corresponds to a debt-to-GDP ratio ($\left(\overline{\lambda}_0 - 1\right)\mu A_h/Y$) slightly above 3. This is about twice the debt-to-GDP ratio in the US in recent years (1.52) reported in Guvenen et al. (2023), confirming that the heterogeneous-return equilibrium allows substantial amounts of borrowing in the model.

Although $\lambda < \overline{\lambda}_0$ holds τ_a at zero, we can use Assumption 2 to find the maximum wealth tax ($\bar{\tau}_a$) that sustains the equilibrium for a given λ. So, as an alternative experiment, we choose λ to generate a debt-to-GDP ratio of 1.5 to match the US economy and plot the corresponding $\bar{\tau}_a$ as we vary β and z_ℓ/z_h in Figure F.3 (left panel) in Appendix F. For $\beta = 0.8$ and $z_\ell/z_h = 0.3$, a wealth tax up to 4% can be sustained, and for $\beta = 0.97$ and $z_\ell/z_h = 0.5$,a wealth tax above 6% can be sustained. The right panel shows the corresponding return heterogeneity ($R_h - R_\ell$) in each case, which ranges from 3% to 6%.

To sum up, these comparisons show that the model we analyze has a unique

[17]Other parameter values are set as follows: Average life expectancy is 50 years ($\delta = 49/50$); the share of H-type entrepreneurs is $\mu = 0.1$; the capital intensity is $\alpha = 0.4$; and the capital income tax is $\tau_k = 25\%$.

heterogeneous-return equilibrium that allows substantial borrowing and can sustain high wealth tax rates under a wide range of plausible parameter values.

3.3 Stationary Wealth Distribution

The following lemma characterizes the gross saving rates of entrepreneurs.

Lemma 2. ***(Saving and Dissaving in the Stationary Equilibrium)*** *In the stationary heterogeneous-return equilibrium, the rates of return of the L-type and the H-type satisfy the following inequalities: $\beta\delta R_\ell < 1 < \beta\delta R_h < 1/\delta$. As a result, the wealth of the L-type (H-type) shrinks (grows) with age. Therefore, the H-type is wealthier than the L-type: $s_h > \mu$.*

Combining Lemma 2 with Proposition 2 shows that $\mu < s_h < 1/\lambda$.

Using this Lemma, we now derive the stationary wealth distribution and show a key property. Since both types start life with $\bar{a}$, and L-type's wealth shrinks while H-type's wealth grows, each group's wealth distribution lies in two non-overlapping (except at $\bar{a}$) intervals: $(0, \overline{a}]$ and $[\overline{a}, \infty)$, with (endogenously-determined) discrete mass points: $\left\{\ldots, (\beta\delta R_\ell)^2 \overline{a}, \beta\delta R_\ell \overline{a}, \overline{a}\right\}$ and $\left\{\overline{a}, \beta\delta R_h \overline{a}, (\beta\delta R_h)^2 \overline{a}, \ldots\right\}$. The population share at wealth level $(\beta\delta R_i)^t \overline{a}$ is equal to the fraction of each type who has lived exactly t years. So, the wealth distribution has a geometric distribution with parameter δ :[18]

$$\Gamma_i \left((\beta\delta R_i)^t \overline{a}\right) = \Pr(z = z_i) \Pr(\text{age} = t) = \Pr(z = z_i)\, \delta^t (1 - \delta). \tag{36}$$

This structure allows us to define a measure of wealth concentration at the top, by dividing the total wealth of H-type entrepreneurs older than age t :

$$A_{h,t} \equiv (1 - \delta) \sum_{s=t}^{\infty} \left(\beta\delta^2 R_h\right)^s \mu\overline{a} = \left(\beta\delta^2 R_h\right)^t \mu A_h, \tag{37}$$

by aggregate wealth. Hence, the wealth share of the top x percent is:

$$s(x) \equiv \frac{\left(\beta\delta^2 R_h\right)^{t(x)} \mu A_h}{K} = \left(\beta\delta^2 R_h\right)^{t(x)} s_h, \tag{38}$$

where $t(x)$ corresponds to the age above which agents are in the top x percent of the wealth

[18] This characterization follows Jones (2015), adapted to the discrete time setting. We discuss the distribution and its response to changes in the environment in Appendix G.

distribution.[19] It is easy to see from (38) that the wealth distribution has a Pareto right tail, which is one of the most salient features of the wealth distribution in modern economies (Vermeulen, 2016). To see this, let $S(x) \equiv s(x)/s(10x)$ be the share of the wealth held by the top $10x$ percent that is held by the top x percent. From (38), this is:

$$S(x) = \left(\beta\delta^2 R_h\right)^{-\frac{\log 10}{\log \delta}}, \tag{39}$$

which is independent of x and increasing in the returns of high-productivity entrepreneurs. Hence, the distribution is Pareto, with the *inverse* of the tail index given by $\eta = -\log(\beta\delta^2 R_h)/\log(\delta)$. Using Lemma 2, η satisfies $0 < \eta < 1$ and increases (inequality is higher) with R_h as expected.[20]

4 Effects of Wealth Taxation

In this section, we consider the effects of increasing the wealth tax on equilibrium outcomes. The results here are global in nature—they hold for any starting level of $\tau_a < \overline{\tau}_a$. We abstract from other taxes to focus on the trade-offs between these two forms of capital taxation. In Section 4.1, we do not impose a government budget constraint; in Section 4.2, we do. In Section 5, we turn our attention to the optimal combination of capital income and wealth taxes that maximizes average newborn welfare.

4.1 Effects on Aggregate Productivity and Returns

Because τ_k does not affect Z, we can study the effect of τ_a on Z without needing to specify the government budget. We now show our first main result of the paper: An increase in the wealth tax increases aggregate productivity, Z.

Proposition 3. ***(Efficiency Gains from Wealth Taxation)*** *For all $\tau_a < \overline{\tau}_a$, a higher wealth tax increases the steady-state aggregate productivity level, $\frac{dZ}{d\tau_a} > 0$.*

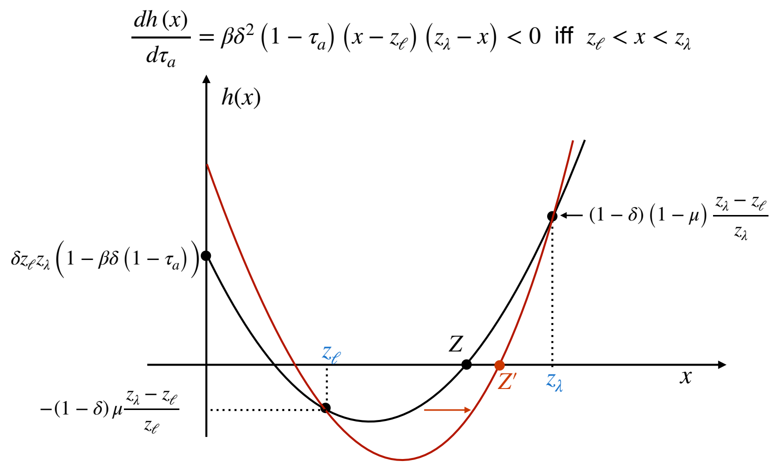
The proof involves showing that the quadratic equation (31) discussed above that pins down Z shifts down and to the right when τ_a is raised, as shown in Figure 2.[21]

[19] Formally, the wealth share of the top x percent corresponds to the wealth share of H-type entrepreneurs of age $t = \log x/\log\delta$.

[20] Although, technically speaking, the Pareto distribution is continuous, Γ can be thought of a discrete counterpart, with the same fractal property as the Pareto.

[21] The sketch of the proof is easy to see diagrammatically: the y-intercept of the polynomial h in Figure 2 is given by $\delta z_\ell z_\lambda(1 - \beta\delta(1 - \tau_a))$, so it increases with τ_a, and the values of the parabola are fixed at z_ℓ

Figure 2: Efficiency Gains from Wealth Taxation



Note: The figure plots the quadratic polynomial on the left hand side of equation (31) for τ_a (black line) and $\tau_a' > \tau_a$ (red line). The equilibrium productivity levels are given by the larger root, marked with the two circles on the horizontal axis.

Recall that $Z \equiv s_h z_\lambda + (1 - s_h) z_\ell$, so a higher Z must follow from an increase in s_h—that is, from the reallocation of wealth towards the H-type, which means higher wealth inequality. We state this as a corollary given its substantive importance for later results.

Corollary 1. ***(Wealth Taxation Increases Wealth Inequality)*** *For all $\tau_a < \overline{\tau}_a$, a higher τ_a reallocates wealth towards the H-type, $ds_h/d\tau_a > 0$, which increases wealth inequality (because $s_h > \mu$ from Lemma 2).*

This result has important implications for returns. The rise in wealth inequality can only happen if τ_a increases the return gap $\log R_h - \log R_\ell$, since the savings decision is linear and only depends on after-tax returns. However, the effect of wealth taxes on the return gap is, in principle, ambiguous. The gap increases due to the use-it-or-lose-it effect of wealth taxes (Proposition 1) but it decreases due to the effect of taxes on productivity (Proposition 3).[22] We can nevertheless determine the effect of wealth taxes on returns in steady state by exploiting the relationship between returns and wealth inequality implied by Equation (29):

$$R_h = \frac{1}{\beta\delta^2}\left(1 - \frac{(1-\delta)\mu}{s_h}\right) \quad \text{and} \quad R_\ell = \frac{1}{\beta\delta^2}\left(1 - \frac{(1-\delta)(1-\mu)}{(1-s_h)}\right). \tag{40}$$

and z_λ (at values shown on the figure), forcing the x-intercepts (the roots of h) to shift down and to the right.

[22] This is easier to see by noting that $R_h - R_\ell = \left(\frac{1}{\beta\delta} - (1-\tau_a)\right)\frac{(z_\lambda - z_\ell)}{Z}$.

These two equations imply that a change in productivity (and hence on wealth inequality) is necessarily accompanied by an increase in the return gap, so that the use-it-or-lose-it effect always dominates. In fact, they show a stronger version of this result, which will become useful later: increasing τ_a increases R_h and *reduces* R_ℓ. It also reduces the population-weighted returns both in levels and in logs. The next proposition formalizes these results.

Proposition 4. ***(Wealth Tax Increases Equilibrium Dispersion of Returns)*** *For all $\tau_a < \overline{\tau}_a$, an increase in the wealth tax increases the rate of return of the H-type and reduces that of the L-type. That is,*

$$\frac{d\log R_h}{d\tau_a} \equiv \xi_{\tau_a}^{R_h} = \xi_Z^{R_h} \times \frac{d\log Z}{d\tau_a} > 0 \qquad \text{and} \qquad \frac{d\log R_\ell}{d\tau_a} \equiv \xi_{\tau_a}^{R_\ell} = \xi_Z^{R_\ell} \times \frac{d\log Z}{d\tau_a} < 0, \quad (41)$$

where $\xi_Z^{R_i} = \frac{d\log R_i}{d\log Z}$. Furthermore, the population-weighted average of returns and log returns decline with τ_a:

$$\frac{d\left(\mu R_\ell + (1-\mu) R_h\right)}{d\tau_a} < 0; \quad (42)$$

$$\frac{d\left(\mu \log R_h + (1-\mu)\log R_\ell\right)}{d\tau_a} = \left(\mu\xi_Z^{R_h} + (1-\mu)\,\xi_Z^{R_\ell}\right)\frac{d\log Z}{d\tau_a} < 0. \quad (43)$$

Equation (43) means that the average elasticity of returns with respect to productivity is negative. This result will be important in shaping the welfare consequences of the wealth tax in Section 5. This proposition also highlights an important property of the effects of wealth taxes: They can be characterized indirectly through the effect of wealth taxes on productivity. We use this insight below as we proceed to determine the effects of τ_a on aggregate variables and welfare.

4.2 Effects on Aggregate Variables

We now turn to the response of aggregate variables—like capital, output, and wages—to changes in the wealth tax. To do this, we need to specify the government budget and how it is balanced with the capital income tax because K adjusts in the stationary equilibrium according to equation (26) in response to changes in τ_k.

Government budget. The government uses the revenues collected from the capital income and wealth taxes to finance (unproductive) government expenditures G and lump-sum transfers to workers T:

$$G + T \; = \tau_k \alpha Y + \tau_a K \; = \; \left(\tau_k + \tau_a \frac{\beta\delta\left(1-\tau_k\right)}{1-\beta\delta\left(1-\tau_a\right)} \right) \alpha Y. \tag{44}$$

What should we assume about total government spending, $G+T$? We consider two assumptions. The first one has empirical appeal and greatly simplifies the analysis: we assume that both G and T are fixed fractions of output. The second one is to assume that $G+T$ is constant—independent of the size of the economy. The first assumption implies that spending will rise/fall linearly with output if different taxes affect the output level, whereas the second assumption implies that the tax experiments we consider are revenue neutral. We start with the first assumption.

Assumption 3. ***(Constant Government Spending Share)*** *Assume that $G = \theta_G \alpha Y$ and $T = \theta_T \alpha Y$, so total tax revenue is $G + T = \theta \alpha Y$, where $\theta \equiv \theta_G + \theta_T$.*

Under Assumption 3, equation (44) implies a tight link between τ_k and τ_a:[23]

$$\frac{1-\theta}{1-\beta\delta} = \frac{1-\tau_k}{1-\beta\delta\left(1-\tau_a\right)}. \tag{45}$$

A special case worth highlighting is when $\theta = 0$: there are no revenue requirements, so taxation only serves to redistribute *among entrepreneurs* or to increase productivity. In this case, it must be that either $\tau_k \geq 0$ and $\tau_a \leq 0$ or $\tau_k \leq 0$ and $\tau_a \geq 0$, with no taxation also being feasible, $\tau_k = \tau_a = 0$.

We first show that under Assumption 3, all aggregate quantities—capital, output, and wages—increase when the wealth tax is raised and the capital income tax is reduced accordingly to keep the government budget balanced. Since Proposition 3 established that Z rises with τ_a, all we need to show is that aggregates increase with Z. Notice that this increase in productivity and aggregates happens even as the total tax revenue collected increases as per Assumption 3, and regardless of how the revenue is spent on G versus T. (As we will see in a moment, aggregates would increase even more if we instead assume revenue neutrality.)[24] These results are summarized in the following lemma.

[23] $\tau_k = \theta$ without a wealth tax ($\tau_a = 0$) and $\tau_a = \frac{\theta(1-\beta)}{\beta(1-\theta)}$ without a capital income tax ($\tau_k = 0$).

[24] Note that if τ_a were raised without any change in τ_k, aggregate capital would have decreased, because of the negative (and empirically probably large) elasticity of capital with respect to τ_a, discussed above and shown in Figure F.2a.

Lemma 3. *For all $\tau_a < \overline{\tau}_a$, under Assumption 3, the steady-state level of capital is*

$$K = \left(\alpha \frac{\beta\delta(1-\theta)}{1-\beta\delta}\right)^{\frac{1}{1-\alpha}} Z^{\frac{\alpha}{1-\alpha}} L, \tag{46}$$

and the long run elasticities of aggregate variables with respect to productivity are

$$\xi_Z^K = \xi_Z^Y = \xi_Z^w = \frac{\alpha}{1-\alpha}, \tag{47}$$

where $\xi_Z^x \equiv \frac{d\log x}{d\log Z}$. Moreover, the wealth level of each entrepreneurial type is

$$A_h = \frac{1}{\mu}\frac{Z - z_\ell}{z_\lambda - z_\ell}K \qquad \frac{dA_h}{dZ} \propto Z^{\frac{2\alpha-1}{1-\alpha}}\left(Z - \alpha z_\ell\right) > 0 \tag{48}$$

$$A_\ell = \frac{1}{1-\mu}\frac{z_\lambda - Z}{z_\lambda - z_\ell}K \qquad \frac{dA_\ell}{dZ} \propto Z^{\frac{2\alpha-1}{1-\alpha}}\left(\alpha z_\lambda - Z\right) \lesseqgtr 0, \tag{49}$$

if and only if $\alpha z_\lambda \lesseqgtr Z$.

Two remarks are in order. First, equation (46) implies that the steady-state levels of capital, output, and wages respond to the wealth tax *only* through its effect on aggregate productivity. Thus,

$$\xi_{\tau_a}^K = \xi_{\tau_a}^Y = \xi_{\tau_a}^w = \frac{\alpha}{1-\alpha} \times \frac{d\log Z}{d\tau_a}. \tag{50}$$

Second, whereas A_h increases unambiguously with the wealth tax, A_ℓ decreases unless $\alpha z_\lambda \geq Z$ (as low-productivity entrepreneurs are born with higher wealth but also dissave at a higher rate). This condition provides a threshold for α because the equilibrium Z is independent of α (see eq. 31).

Let us now consider what happens if, instead of Assumption 3, we assume that the total tax revenue is constant: $G + T = \overline{\theta}$. In this case, because tax revenue does not rise with output, the same rise in τ_a is matched with a larger decline in τ_k than under 3, implying a stronger positive response of aggregates to τ_a. The following lemma states this result.

Lemma 4. *Assume that total government is fixed, $G+T = \overline{\theta}$. Then, the (semi-)elasticities of capital, output, and wages to a change in the wealth tax satisfy*

$$\xi_{\tau_a}^K = \xi_{\tau_a}^Y = \xi_{\tau_a}^w > \frac{\alpha}{1-\alpha}\frac{d\log Z}{d\tau_a}. \tag{51}$$

4.3 Effects on Individual Welfare

To understand the welfare effects, we start with the value of workers,

$$V_w = \frac{1}{1-\beta\delta}\log\left(w+T\right), \tag{52}$$

and the value of an entrepreneur of type $i \in \{h, \ell\}$ with assets a:

$$V_i\left(a\right) = \frac{1}{1-\beta\delta}\log\left(a\right) + \frac{1}{\left(1-\beta\delta\right)^2}\left[\log\left(\beta\delta\right)^{\beta\delta}\left(1-\beta\delta\right)^{1-\beta\delta} + \log R_i\right], \tag{53}$$

which is obtained by substituting the solution of the entrepreneurs' problem into (10). This gives rise to the following result characterizing the conditions for welfare changes after an increase in the wealth tax.

Proposition 5. *For all $\tau_a < \overline{\tau}_a$, under Assumption 3, an increase in the wealth tax increases the welfare of workers and newborn H-type entrepreneurs,*

$$\frac{dV_w}{d\tau_a} > 0 \qquad and \qquad \frac{dV_h\left(\overline{a}\right)}{d\tau_a} > 0. \tag{54}$$

Moreover, an increase in the wealth tax increases the welfare of newborn L-type entrepreneurs and the ex ante welfare of newborn entrepreneurs under the following conditions:

$$\frac{dV_\ell\left(\overline{a}\right)}{d\tau_a} > 0 \quad if \quad \xi_Z^K > \frac{-1}{1-\beta\delta}\xi_Z^{R_\ell}; \tag{55}$$

$$\frac{d\left(\mu V_h\left(\overline{a}\right) + \left(1-\mu\right)V_\ell\left(\overline{a}\right)\right)}{d\tau_a} > 0 \quad if \quad \xi_Z^K > \frac{-1}{1-\beta\delta}\left(\mu\xi_Z^{R_h} + \left(1-\mu\right)\xi_Z^{R_\ell}\right). \tag{56}$$

An increase in the wealth tax increases the welfare of workers and high-productivity entrepreneurs, because it increases wages, transfers, average wealth, and the returns of high-productivity entrepreneurs.[25] The welfare change of low-productivity entrepreneurs and entrepreneurs as a group is ambiguous because of two countervailing forces that are apparent in (53): a higher wealth tax increases the initial wealth of entrepreneurs, Lemma 3, but decreases the returns of low-productivity entrepreneurs as well as the average returns of entrepreneurs, Proposition 4.

[25] It is worth noting that, under Assumption 3, a worker's the total income is $w+T = \left(\left(1-\alpha\right)+\theta_T\alpha\right)Y/L$ with (semi-)elasticity with respect to the wealth tax given by $\xi_{\tau_a}^{w+T} \equiv d\log\left(w+T\right)/d\tau_a = \frac{\alpha}{1-\alpha} \times d\log Z/d\tau_a$.

If the pass-through of productivity to capital is sufficiently high, that is, if α is sufficiently high, entrepreneurs overall benefit from the increase in the wealth tax, despite the decrease in returns. In this way, the conditions established in the Proposition 5 imply threshold values for α above which the welfare of entrepreneurs increases. However, for a range of plausible parameter values, these thresholds turn out to be too high, with α having to be above 0.7. See Figure F.4 in Appendix F.

5 Optimal Taxation

The government's objective is to maximize the equilibrium utilitarian welfare of the newborns, $\mathcal{W}$, by choosing the optimal combination of capital income and wealth taxes, subject to its budget constraint. Let $n_w \equiv {}^{L}/_{(1+L)}$ represent the fraction of workers in the population. The government's problem is

$$\max_{\tau_k,\tau_a} \mathcal{W} \equiv n_w V_w(w) + (1-n_w)\left(\mu V_h(\overline{a}) + (1-\mu) V_\ell(\overline{a})\right) \qquad \text{s.t. } (44), \tag{57}$$

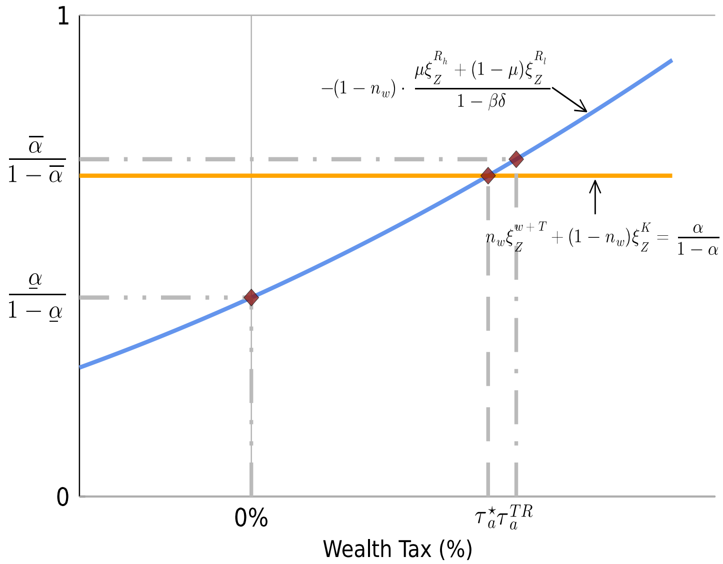
We can make the trade-off faced by the government clearer by substituting in the value functions of workers and entrepreneurs from (52) and (53):

$$\mathcal{W} = \frac{1}{1-\beta\delta}\left(n_w \log(w+T) + (1-n_w)\log\overline{a}\right) + \frac{1-n_w}{(1-\beta\delta)^2}\left(\mu \log R_h + (1-\mu)\log R_\ell\right) + v. \tag{58}$$

where $v \equiv \frac{1-n_w}{(1-\beta\delta)^2}\log(\beta\delta)^{\beta\delta}(1-\beta\delta)^{1-\beta\delta}$ is a constant. Increasing the wealth tax (while simultaneously reducing the capital income tax as in 45) affects aggregates through its effect on aggregate productivity and leads to higher wages and wealth (Lemma 3). We call this the level effect of wealth taxation. However, increasing the wealth tax also results in lower average log returns (Proposition 4). This is the return dispersion effect. An interior solution balances these effects and satisfies $d\mathcal{W}/d\tau_a = 0$, where $d\mathcal{W}/d\tau_a$ depends on the elasticities of aggregates with respect to Z.

Figure 3 illustrates the forces at play. The elasticities of workers' income and wealth with respect to productivity $\left(n_w \xi_Z^{w+T} + (1-n_w)\xi_Z^K\right)$ give the (percentage) gain in workers' and entrepreneurs' welfare as the wealth tax increases (raising productivity). These elasticities are constant in our economy as shown in Lemma 3, and are both equal to $\alpha/(1-\alpha)$. At the same time, the (negative) average elasticity of returns $\left(\mu \xi_Z^{R_h}(\tau_a) + (1-\mu)\xi_Z^{R_\ell}(\tau_a)\right)$ is increasing in τ_a, reflecting the widening gap between low and high returns as the wealth tax

Figure 3: Determination of the Optimal Wealth Tax



Note: The figure shows the conditions satisfied by the optimal wealth tax solving (57). The horizontal line is the (population) average of the elasticity of workers' income and capital with respect to productivity, ξ_Z^{w+T} and ξ_Z^K respectively. The increasing line is proportional to the negative of the average elasticity of returns with respect to productivity $\left(\xi_Z^R\right)$. The optimal wealth tax is denoted by $\tau_a^\star$, and $\tau_a^{TR}=\frac{\theta(1-\beta\delta)}{\beta\delta(1-\theta)}$ denotes the tax reform tax, the level at which $\tau_k=0$. The remaining parameters are as follows: $\delta={}^{49}/_{50}$, $\beta\delta=0.96$, $\mu=0.10$, $z_h=1$, $\theta=25\%$, and $\alpha=0.4$.

increases. The intersection of the two lines pins down the optimal wealth tax. We formalize this in the following proposition.

Proposition 6. ***(Optimal Taxes)*** *Under Assumption 3, there is a unique combination of tax instruments $(\tau_a^\star,\tau_k^\star)$ that maximizes the utilitarian welfare. An interior solution $\tau_a^\star<\overline{\tau}_a$ is the solution to:*

$$0=\left[\underbrace{n_w\xi_Z^{w+T}+(1-n_w)\,\xi_Z^K}_{\textit{Level Effect }(+)}+\frac{1-n_w}{1-\beta\delta}\left(\underbrace{\mu\xi_Z^{R_h}(\tau_a)+(1-\mu)\,\xi_Z^{R_\ell}(\tau_a)}_{\textit{Return Dispersion }(-)}\right)\right]\underbrace{\frac{d\log Z}{d\tau_a}}_{(+)}. \tag{59}$$

Furthermore, there are two cutoff values for α, $\underline{\alpha}$ and $\overline{\alpha}$, such that $(\tau_a^\star,\tau_k^\star)$ has the following *properties:*

$$\tau_a^\star \in \left[1 - \frac{1}{\beta\delta}, 0\right) \text{ and } \tau_k^\star > \theta \qquad \text{if } \alpha < \underline{\alpha}$$

$$\tau_a^\star \in \left[0, \frac{\theta(1-\beta\delta)}{\beta\delta(1-\theta)}\right] \text{ and } \tau_k^\star \in [0, \theta] \qquad \text{if } \underline{\alpha} \le \alpha \le \bar{\alpha}$$

$$\tau_a^\star \in \left(\frac{\theta(1-\beta\delta)}{\beta\delta(1-\theta)}, \tau_a^{\max}\right) \text{ and } \tau_k^\star < 0, \qquad \text{if } \alpha > \bar{\alpha}$$

where $\tau_a^{\max} \ge 1$*,* $\underline{\alpha}$ *and* $\overline{\alpha}$ *are the solutions to equation (59) with* $\tau_a = 0$ *and* $\tau_a = \frac{\theta(1-\beta\delta)}{\beta\delta(1-\theta)}$*, respectively. When* $\theta = 0$ *and there are no revenue needs, so* $\underline{\alpha} = \overline{\alpha}$*.*

Figure 3 also clarifies the roles of the thresholds $\underline{\alpha}$ and $\overline{\alpha}$.[26] The lower threshold $\underline{\alpha}$ marks the level of $n_w \xi_Z^{w+T} + (1-n_w)\xi_Z^K$ for which $\tau_a = 0$ is optimal. Any $\alpha > \underline{\alpha}$ implies a higher scope for workers' income and capital to rise with the wealth tax and thus a positive optimal wealth tax. Relative to a benchmark with $\tau_a = 0$, capital income taxes are lower and welfare gains come from the level effect on wages, transfers, and capital. At the same time, the dispersion in returns is higher and, as a result, entrepreneurs' welfare is lower than without the wealth tax. The upper threshold $\overline{\alpha}$ is similarly defined by the level of $n_w \xi_Z^{w+T} + (1-n_w)\xi_Z^K$ for which

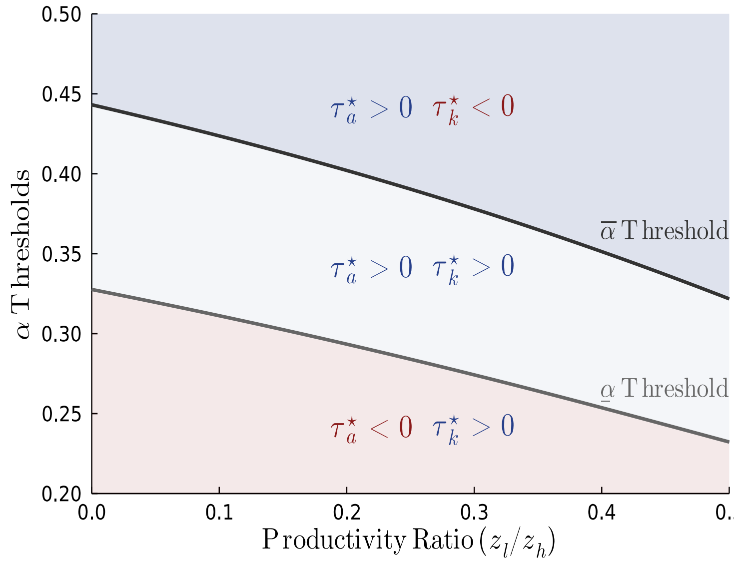
$$\tau_a = \tau_a^{TR} \equiv \frac{\theta(1-\beta\delta)}{\beta\delta(1-\theta)} \tag{60}$$

is optimal. At that level, the wealth tax finances all government spending, so $\tau_k = 0$. Consequently, any $\alpha > \overline{\alpha}$ implies that the optimal tax combination is one of a positive wealth tax and a capital income subsidy. Finally, the upper bound on the wealth tax ($\tau_a^{\max}$) ensures that R_ℓ remains positive.

Figure 4 shows how the thresholds for α vary with the dispersion in entrepreneurial productivity when we set $\theta = 0.25$, implying a capital income tax rate of 25% in the absence of wealth taxes. Both thresholds decline as the dispersion of productivity decreases and maintain a gap of about 0.1 that includes the typical values of α used in the literature, between 0.3 and 0.4. For example, a value of α of $1/3$ is always in the intermediate range, implying positive values for the optimal levels of capital income and wealth taxes, while a value of α of 0.4 implies a positive wealth tax and a capital income subsidy if the ratio of productivities satisfies $z_\ell / z_h \ge 0.2$.

[26] The value of the thresholds depend on Z, which is endogenous but independent of α (equation 31).

Figure 4: α Thresholds for the Optimal Wealth and Capital Income Taxes



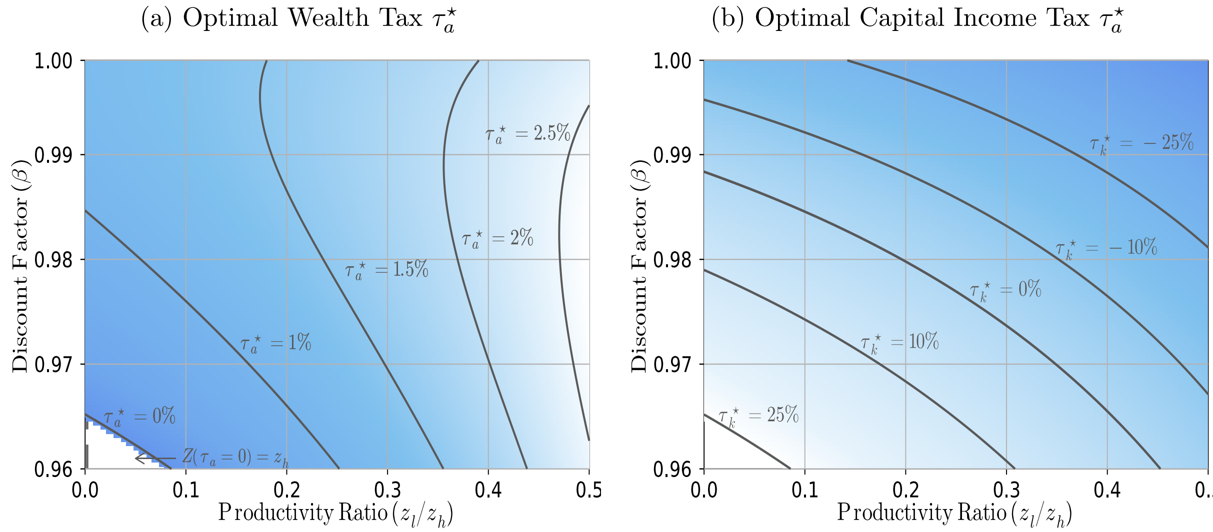
Note: The figure reports the threshold values of α for the optimal wealth tax to be positive (lower threshold) and capital income taxes to be positive (upper threshold) for different levels of productivity dispersion (z_ℓ/z_h). We set the remaining parameters as follows: $\delta = 49/50$, $\beta\delta = 0.96$ $\mu = 0.10$, $z_h = 1$, $\theta = 25\%$, and $\alpha = 0.4$.

Figure 5 shows the levels of the optimal wealth and capital income taxes for different combination of parameters, holding fixed α at 0.4. The optimal wealth tax is positive except for corner cases where the condition in Assumption 2 for the heterogeneous return equilibrium imply a negative upper bound for wealth taxes ($\overline{\tau}_a < 0$). This happens when there is no capital misallocation because L-type entrepreneurs are either too unproductive or too impatient, leading wealth to be concentrated in the hands of H-type entrepreneurs. As the dispersion of productivity decreases, or entrepreneurs are more patient, there is more misallocation and a higher optimal level of wealth tax in the range of 0 to 2 percent for most parameter combinations. The flip side of this pattern is the decrease in the optimal level of capital income taxes that eventually become subsidies as the optimal wealth tax increases.

6 Wealth Taxation with Innovation

We now extend the baseline model to endogenize the *distribution of entrepreneurial productivity*, μ, as the outcome of a costly and risky innovation process and to determine how innovation depends on the combination of capital income and wealth taxes. Specifically, we assume that newborn innovators come up with new ideas for production. The quality

Figure 5: Optimal Taxes



Note: The figures report the value of the optimal wealth and capital income taxes for combinations of the discount factor (β) and productivity dispersion (z_ℓ/z_h). We set the remaining parameters as follows: $\delta = 49/50$, $\mu = 0.10$, $z_h = 1$, $\theta = 25\%$, and $\alpha = 0.4$.

of these ideas is captured by the productivity, z, of the technology they describe. Once an idea is generated, the innovator uses it to produce and has access to it for the rest of their lifetime (akin to having a perpetual patent), just as in Section 2.

Innovation requires costly effort. Crucially, an innovator's effort is not guaranteed to grant them success, as innovation is a risky endeavor. Instead, effort, e, determines the probability that the innovator's idea turns into a high-productivity technology, $\Pr(z = z_h) = p(e)$. The innovators' problem is

$$\max_e \; p(e) V_h(\overline{a}) + (1 - p(e)) V_\ell(\overline{a}) - \frac{1}{(1-\beta\delta)^2} \Lambda(e), \tag{61}$$

where Λ is a strictly increasing, strictly convex, and twice continuously differentiable cost function for effort with $\Lambda(0) = 0$ and $\Lambda'(0) = 0$. The resulting value of an idea corresponds to the value of an entrepreneur with productivity z_h or z_ℓ, depending on the outcome of the innovation process. The value of the entrepreneur is the same as in equation (53). Without loss of generality, we set $p(e) = e$ to simplify the problem.

The optimal effort choice is characterized by the solution to the following equation:[27]

$$\Lambda'(e) = (1-\beta\delta)^2 \left(V_h(\overline{a}) - V_\ell(\overline{a})\right) = \log R_h - \log R_\ell. \tag{62}$$

So, the effort choice depends on the return gap, $\log R_h - \log R_\ell$. A higher return gap generates higher incentives for effort as it captures the difference in lifetime payoffs between high- and low-productivity entrepreneurs. Moreover, because the equilibrium returns are a function of aggregate productivity (equation 40), the optimal effort decision rule is a function of Z and depends on τ_a only through productivity: $e^\star(Z)$.

6.1 Stationary Equilibrium with Innovation

All innovators are identical (at birth) and therefore make the same choices. This makes $\mu^\star \equiv p(e^\star) = e^\star$ the share of high-productivity entrepreneurs. The rest of the economy is characterized as in Section 2. The stationary equilibrium can therefore be stated as a fixed point in μ: Given μ, the equilibrium productivity Z is determined by equation (31). Then, individual innovators take μ, Z, and the implied returns R_h and R_ℓ as given when making their innovation effort choice. This choice, in turn, implies μ.

Definition. (Stationary Equilibrium $\mu^\star$) The equilibrium share of high-productivity entrepreneurs, $\mu^\star$, is the solution to

$$\mu^\star = e^\star(Z(\mu^\star)), \tag{63}$$

where

i. $Z(\mu)$ gives the stationary level of productivity given μ; that is, $Z(\mu)$ is the solution to equation (31), for a given $\mu \in (0,1)$; and

ii. $e^\star(Z)$ gives the optimal innovation effort given Z; that is, $e(Z)$ solves equation (62) for a given $Z \in (z_\ell, z_h]$, where returns are given by equation (40).

A couple of remarks are in order. First, there are no equilibria with innovation in which returns are homogeneous ($s_h > 1/\lambda$). This is because without return dispersion the optimal innovation effort is $e^\star = 0$, as implied by equation (62), making it so that there are no

[27] We assume that the cost function Λ is such that a corner solution is never optimal. This is done by evaluating the equation at $Z \in \{z_\ell, z_h\}$ and ensuring that the solution is interior in both cases.

H-type entrepreneurs. But then there would be no demand for funds coming from high-productivity entrepreneurs ($s_h < 1/\lambda$), leading to a contradiction. Thus, all equilibria must feature return heterogeneity.

Second, the conditions in Assumptions 1 and 2 for the equilibrium to feature return heterogeneity (ensuring that $s_h < 1/\lambda$ holds) must be restated because they depend on μ, which is now endogenous. For instance, as τ_a increases, the return gap grows and entrepreneurs exert more effort, increasing μ and overall productivity, as we show formally below in Propositions 8 and 9. But, as μ increases, so does s_h, making it harder to guarantee that the demand for funds from H-type entrepreneurs is met by the wealth held by the L-types, which is required for the heterogeneous-return equilibrium to arise. This results in a new upper bound for wealth taxes, $\overline{\tau}_a^{\mu} < \overline{\tau}_a$.

We establish the existence of a unique solution to (63), that is a unique fixed point for μ that characterizes the stationary equilibrium of the economy. Existence of the fixed point follows from standard fixed point arguments relying on Cellina's and Brouwer's fixed point theorems (Border, 1985, Thms. 15.1, 16.1). Uniqueness follows from standard comparative statics results for fixed points after showing that the mapping of μ into itself is monotone.[28] We can now state the main result of this section.

Proposition 7. ***(Existence of a Unique Stationary Equilibrium with Innovation)*** *There exists an upper bound for the wealth tax $\overline{\tau}_a^{\mu}$ such that for $\tau_a < \overline{\tau}_a^{\mu}$ there is a unique stationary equilibrium that features heterogeneous returns. That is, there is a unique level of the share of H-type entrepreneurs, $\mu^\star$, such that the optimal level of effort exerted by innovators satisfies $\mu^\star = e^\star(Z(\mu^\star))$, and $Z(\mu^\star) \in (z_\ell, z_h)$ satisfies equation (31). The upper bound for the wealth tax satisfies*

$$\overline{\tau}_a^{\mu} = 1 - \frac{1}{\beta\delta}\left(1 - \frac{1-\delta}{\delta}\frac{1-\lambda\mu^\star(\overline{\tau}_a^{\mu})}{(\lambda-1)\left(1-\frac{z_\ell}{z_h}\right)}\right), \tag{64}$$

where we make the dependence of $\mu^\star$ on τ_a explicit.

6.2 Effect of Wealth Taxes in the Stationary Equilibrium

Innovation. We now show that innovation increases with the wealth tax. The wealth tax increases the *equilibrium* dispersion of returns for any given μ (Proposition 4). This

[28] Specifically, we show that the equilibrium Z is increasing in μ using equation (31) and that the optimal effort $e^\star$ is decreasing in Z because of its effect on return dispersion. We then show that these two results imply that the equilibrium mapping for μ defined in (63) is monotonically decreasing.

increase in return dispersion provides incentives for higher innovation effort, as the returns to a high-productivity idea are higher and the returns to a low-productivity one are lower. The result is an increase in the equilibrium level of innovation effort and hence in the share of H-type entrepreneurs. By contrast, capital income taxes have no effect on equilibrium returns, and hence are neutral for innovation. The proof of this result builds on standard comparative static results for fixed points found in Villas-Boas (1997).

Proposition 8. ***(Innovation Gains from Wealth Taxation)*** *For all $\tau_a < \overline{\tau}_a^{\mu}$, an increase in the wealth tax (τ_a) increases the equilibrium share of high-productivity entrepreneurs, $\mu^{\star}$. Capital income taxes do not affect innovation.*

Productivity. Having established that innovation effort is increasing in the wealth tax, we can also prove that equilibrium productivity increases after taking into account the changes in μ. The proof follows from the fact that the solution to equation (31) is increasing in both μ and τ_a. This ensures that productivity rises with the wealth tax.

Proposition 9. ***(Productivity Gains from Wealth Taxation with Innovation)*** *For all $\tau_a < \overline{\tau}_a^{\mu}$, an increase in the wealth tax (τ_a) increases productivity, $Z^{\star}$.*

Similar to what we found in Section 2, this result implies that the wealth share of H-type entrepreneurs increases with the wealth tax. The fact that the equilibrium level of μ increases implies that the returns gap, $\log R_h - \log R_\ell$, increased as well. We can also show that $dR_\ell/d\tau_a < 0$; however, the direction of the change in R_h cannot be signed without further restrictions on the effort cost function Λ.

Aggregates. Under a balanced budget (Assumption 3), the increase in productivity implies that capital, output, and wages increase in response to an increase in τ_a (and a corresponding reduction in τ_k). This follows directly from Lemma 3, as the steady-state values of these variables do not depend on μ directly.

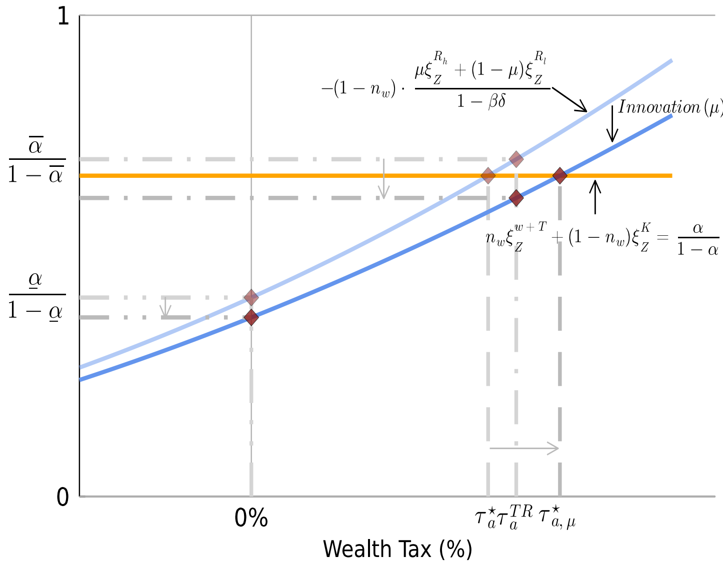
6.3 Optimal Taxes with Innovation

We now turn to the choice of optimal tax rates. As in Section 5, we choose taxes to maximize the welfare of newborns in the stationary equilibrium,

$$\mathcal{W} = n_w V_w + (1-n_w)\left(\mu V_h(\overline{a}) + (1-\mu) V_\ell(\overline{a}) - \frac{\Lambda(\mu)}{(1-\beta\delta)^2}\right), \tag{65}$$

which now includes the cost of innovation effort and the fact that $\mu = e$ in equilibrium.

Figure 6: Optimal Wealth Tax with Endogenous Innovation



Note: The figure shows the conditions satisfied by the optimal wealth tax solving (57) and (66). The horizontal line is the (population) average of the elasticity of workers' income and capital with respect to productivity, ξ_Z^{w+T} and ξ_Z^K, respectively. The increasing lines are proportional to the negative of the average elasticity of returns with respect to productivity $\left(\xi_Z^R\right)$ when μ is fixed and to the elasticity of returns taking into account changes in innovation, the lighter gray line. The optimal wealth tax is denoted by $\tau_{a,\mu}^\star$, the tax reform tax level is $\tau_a^{TR}=\frac{\theta(1-\beta\delta)}{\beta\delta(1-\theta)}$, the level at which $\tau_k=0$. The remaining parameters are as follows: $\delta={}^{49}\!/_{50}$, $\beta\delta=0.96$, $\mu=0.10$, $z_h=1$, $\tau_k=25\%$, and $\alpha=0.4$.

The optimal tax combination is obtained as before, by balancing the increase in welfare coming from the level effect on higher wages and wealth accumulation, with the decrease in returns that accompany the increase in productivity. However, there is now a new margin coming from the change in returns in response to an increase in innovation. We illustrate the effect of incorporating the changes in innovation into the optimal tax choice in Figure 6. The level effect on workers' income and wealth remains the same, but the decrease in returns is not as pronounced as it was with a fixed level of μ. This is because innovation increases average returns and hence implies a higher optimal wealth tax $\tau_{a,\mu}^\star > \tau_a^\star$ (relative to that in Proposition 10).[29] Crucially, this is a change in the level of returns, separate from the change in the population weights, which has no welfare effect, as μ is being chosen optimally by the entrepreneurs. The following proposition formalizes these results.

Proposition 10. *Under Assumption 3, an interior solution* ($\tau_{a,\mu}^\star < \overline{\tau}_a^\mu$) *to the optimal tax combination* $\left(\tau_{a,\mu}^\star,\tau_{k,\mu}^\star\right)$ *that maximizes the newborn welfare,* $\mathcal{W}$, *is the solution to the*

[29] In particular, the sum of the effects of productivity on K, $w+T$, and returns in equation (66) evaluated at $\tau_a^\star$ (the optimum without innovation) is zero because these terms are the same as the ones in equation (59). This leaves the effect of innovation on returns, which is positive, implying that $\tau_{a,\mu}^\star > \tau_a^\star$.

following equation:

$$0 = \Bigg(\underbrace{n_w \xi_Z^{w+T} + (1-n_w)\,\xi_Z^K}_{\textit{Level Effect } (+)} + \underbrace{\frac{1-n_w}{1-\beta\delta}\left(\mu \xi_Z^{R_h} + (1-\mu)\,\xi_Z^{R_\ell}\right)}_{\textit{Reeturn Dispersion } (-)} \Bigg) \frac{d\log Z}{d\tau_a} \tag{66}$$
$$+ \underbrace{\frac{1-n_w}{1-\beta\delta}\left(\mu \xi_\mu^{R_h} + (1-\mu)\,\xi_\mu^{R_\ell}\right)\frac{d\mu}{d\tau_a}}_{\textit{Innovation Effect on Returns } (+)},$$

where $\xi_Z^x \equiv \frac{\partial \log x}{\partial \log Z}$ *is the elasticity of variable* x *with respect to* Z *and* $\xi_\mu^x \equiv \frac{\partial \log x}{\partial \mu}$ *is the (semi-)elasticity with respect to* μ. *Recall from Lemma 3 that* $\xi_Z^{w+T} = \xi_Z^K = \frac{\alpha}{1-\alpha}$.

The sign of the optimal tax rates depends on the strength of the level effect of productivity on wealth and income. Just as in Proposition 10, this implies threshold values for the capital intensity α, $\underline{\alpha}_\mu$ and $\overline{\alpha}_\mu$, such that the optimal wealth tax rate $\tau^\star_{a,\mu}$ is positive if $\alpha > \underline{\alpha}_\mu$ and the capital income tax rate $\tau^\star_{a,\mu}$ is positive if $\alpha < \overline{\alpha}_\mu$. As Figure 6 shows, both these thresholds are lower when innovation is taken into account ($\underline{\alpha}_\mu < \underline{\alpha}$ and $\overline{\alpha}_\mu < \overline{\alpha}$). This can be seen by rewriting equation (66) as

$$\frac{\alpha}{1-\alpha} = -\frac{1-n_w}{1-\beta\delta}\left(\mu \xi_Z^{R_h} + (1-\mu)\,\xi_Z^{R_\ell}\right) - \frac{1-n_w}{1-\beta\delta}\left(\mu \xi_\mu^{R_h} + (1-\mu)\,\xi_\mu^{R_\ell}\right)\frac{d\mu/d\tau_a}{d\log Z/d\tau_a}. \tag{67}$$

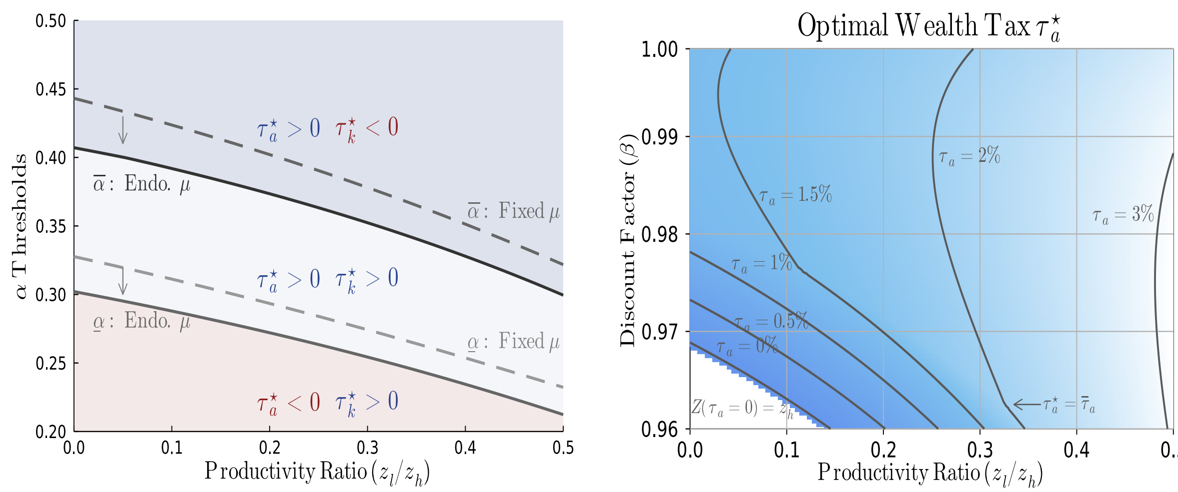
In Section 2, we obtained $\underline{\alpha}$ and $\overline{\alpha}$ by evaluating the first term in this equation at $\tau_a = 0$ or $\tau_k = 0$, respectively. Because the second term is always negative, the corresponding α thresholds with innovation are lower. Figure 7a shows the change in the thresholds for different combinations of parameters.

Figure 7b shows the resulting optimal wealth tax when $\alpha = 0.4$ for different combination of parameters. The additional force in favor of a higher wealth tax brought up by the response of innovation effort actually implies corner solutions for the wealth tax ($\tau_a^\star = \overline{\tau}_a$) when entrepreneurs are impatient or productivity dispersion is high. This explains the kink in the optimal wealth tax curves in the figure. While the magnitudes shown in the figures are meant to be illustrative, they do show that, while the optimal level of the wealth tax rises when innovation is endogenized, it does not imply implausible tax levels, with most parameter combinations implying optimal levels below 2 percent.

Figure 7: Optimal Taxes with Innovation

(a) α Thresholds for Optimal Wealth Taxes

(b) Optimal Wealth Tax



Note: The left panel plots the lower threshold values of α for the optimal wealth tax to be positive and the upper threshold for the capital income tax to be positive for different levels of productivity dispersion (z_ℓ/z_h). The dashed lines correspond to the thresholds $\underline{\alpha}$ and $\overline{\alpha}$ in the benchmark model presented in Figure 4. The right panel plots the values of the optimal wealth tax for combinations of β and z_ℓ/z_h holding α at 0.4. We set the remaining parameters as follows: $\delta = 49/50$, $\mu = 0.10$, $z_h = 1$, and $\theta = 25\%$.

7 Entrepreneurial Effort

We now consider how entrepreneurial effort shapes the *level of productivity* of private enterprises, whereas in the previous section we focused on the endogenous *distribution of productivity*. We show that capital income and wealth taxes have different effects on the effort choice of entrepreneurs. While both taxes affect capital accumulation, capital income taxes directly distort the effort choice of entrepreneurs by reducing the marginal benefit from exerting effort. This introduces a new channel, which we spell out shortly, by which replacing capital income taxes with wealth taxes increases output and welfare.

We introduce effort in a tractable manner that allows us to identify its core implications for wealth and capital income taxation. Effort, e, affects production according to

$$y = (zk)^{\alpha} e^{\gamma} n^{1-\alpha-\gamma}, \tag{68}$$

where $0 \le \gamma < 1 - \alpha$. Exerting effort has a utility cost that we capture by modifying the utility function to

$$u(c, e) = \log(c - h(e)), \tag{69}$$

where $h(e) = \psi e$ and $\psi > 0$.[30] Tractability depends on preserving the constant-returns-to-scale in production and abstracting from income effects in the effort choice as in Greenwood, Hercowitz and Huffman (1988).[31] Together, these properties allow us to solve the model analytically, as we show in Appendix D.

The solution of the problem with entrepreneurial effort inherits the properties of our benchmark model after a suitable change of variables. We define consumption net of effort costs as $\hat{c} = c - h(e)$ and write the problem as

$$V(a, z) = \max_{a', \hat{c}, e} \log(\hat{c}) + \beta\delta V(a', z) \tag{70}$$
$$\text{s.t. } \hat{c} + a' = \underbrace{(1 - \tau_a) + (1 - \tau_k)\left(r + \frac{\hat{\pi}(z, k, e)}{a}\right)}_{\hat{R}(z)} a,$$

where $\hat{\pi}$ stands for profits net of effort costs:

$$\hat{\pi}(z, k, e) = \max_n y - wn - rk - \frac{1}{1 - \tau_k} h(e). \tag{71}$$

Crucially, the capital income tax has a direct effect on effort because effort costs are paid privately by the entrepreneur. Thus, the effective marginal cost of effort is $\frac{h'(e)}{1-\tau_k}$. This changes the relationship between capital income taxes and aggregate output, capital, and wages, but does not change the equilibrium behavior of aggregate productivity (equation 31). This is because the capital level adjusts in steady state so that the after-tax return net of effort costs satisfies

$$\hat{R}(z) = (1 - \tau_a) + \left(\frac{1}{\beta\delta} - (1 - \tau_a)\right)\frac{z}{Z} \tag{72}$$

just as in Lemma 1, preserving the neutrality of τ_k for returns and productivity. Consequently, the results of our benchmark model regarding the existence of a stationary competitive equilibrium and the efficiency gains from wealth taxation (Propositions 2 and 3) remain unchanged, as do the effects of wealth taxes on after tax returns.

Proposition 11. *A stationary competitive equilibrium exists and is unique if and only if* λ

[30] In general, we can let effort affect production according to an increasing function $g(e)$, and we only require that the ratio $h'(e)/g'(e)$ is constant. See Appendix D.

[31] Abstracting from income effects leads to an overstatement of the response of effort to taxation, as wealthier entrepreneurs may want to exert less effort in the presence of income effects.

satisfies Assumption 1, and an increase in the wealth tax in such an equilibrium increases productivity Z.

Focusing now on the effect of taxes on aggregate variables, we obtain closed-form expressions for equilibrium quantities as a function of aggregate capital, K, and productivity, Z, paralleling the results of Lemma 1. The main difference is, of course, the introduction of effort. Aggregate effort is

$$E = \left(\frac{(1-\tau_k)\,\gamma}{\psi}\right)^{\frac{1}{1-\gamma}} (ZK)^{\frac{\alpha}{1-\gamma}}\, L^{\frac{1-\alpha-\gamma}{1-\gamma}}. \tag{73}$$

Effort is disincentivized by capital income taxes, that reduce its after-tax marginal product, effectively making effort more costly. Consequently, capital income taxes also reduce aggregate output and wages,

$$Y = \left(\frac{(1-\tau_k)\,\gamma}{\psi}\right)^{\frac{\gamma}{1-\gamma}} (ZK)^{\frac{\alpha}{1-\gamma}}\, L^{\frac{1-\alpha-\gamma}{1-\gamma}}, \tag{74}$$

$$w = (1-\alpha-\gamma)\left(\frac{(1-\tau_k)\,\gamma}{\psi}\right)^{\frac{\gamma}{1-\gamma}} \left(\frac{ZK}{L}\right)^{\frac{\alpha}{1-\gamma}}. \tag{75}$$

By contrast, wealth taxes do not directly affect the effort choice because they do not affect the fraction of profits retained by the entrepreneur and affect production and wages only through their effect on aggregate productivity.

The effect of capital income taxes on entrepreneurial effort introduces a new channel affecting the optimal tax combination: When the government balances its budget, a higher wealth tax reduces the level of the capital income tax, incentivizing entrepreneurial effort and, through it, increasing aggregate output, capital, and wages, as equations (73) to (75) make clear. This works on top of the increase in capital, output, and wages coming from increased productivity, described in Lemma 3. Put in terms of the trade-off described in Section 5, the presence of entrepreneurial effort strengthens the (positive) level effect of wealth taxes without changing the return dispersion effect (because of the neutrality of capital income taxes for after-tax returns). The result is an optimal tax combination that now involves a higher wealth tax and a lower capital income tax.

Proposition 12. *Under Assumption 3, there exists a unique tax combination* $\left(\tau^\star_{a,e}, \tau^\star_{k,e}\right)$

that maximizes the utilitarian newborn welfare. An interior solution $\tau^{\star}_{a,e} < \overline{\tau}_a$ satisfies

$$\frac{\gamma}{1-\alpha-\gamma}\frac{\beta\delta}{\frac{\log Z}{d\tau_a}} + \frac{\alpha}{1-\alpha-\gamma} = -\frac{1-n_w}{1-\beta\delta}\left(\mu\xi_Z^{\hat{R}_h} + (1-\mu)\,\xi_Z^{\hat{R}_\ell}\right), \tag{76}$$

where $\xi_Z^x \equiv \frac{d\log x}{d\log Z}$ is the elasticity of variable x with respect to Z. Moreover, $\tau^{\star}_{a,e} > \tau^{\star}_a$, with $\tau^{\star}_a$ being the optimal tax described in Proposition 6.

8 Persistence of Entrepreneurial Productivity

Finally, we consider the role of *fluctuations of entrepreneurial productivity* in shaping our results on productivity and welfare gains from wealth taxation. The models studied so far assume that entrepreneurs have the same productivity throughout their lives. In them, an increase in the wealth tax benefits H-type entrepreneurs whose returns and wealth increase permanently, reducing misallocation. However, fluctuations in individual entrepreneurial productivity increase misallocation as wealthy (formerly productive) entrepreneurs lose their productivity. Nevertheless, we show that entrepreneurial productivity needs only to be persistent (i.e., positively autocorrelated) in order to preserve our main results.

To study the role of productivity persistence, we put forth a model where entrepreneurial productivity follows a Markov process and entrepreneurs are infinitely lived. This model remains tractable while allowing for fluctuations in individual productivity, and provides a clear cut answer to the conditions under which wealth taxes increase productivity and welfare. As in Section 2, there are two types of agents, homogeneous workers of size L and heterogenous entrepreneurs of size 1, but they are now infinitely-lived. This amounts to setting $\delta = 1$.[32] Preferences are as in Section 2, as is the behavior of workers. The entrepreneurs' production problem is given by (4). Thus, we can aggregate as in Lemma 1. We provide a summary of the model and the results here and a detailed derivation of results in Appendix E.

The main change comes from having entrepreneurial productivity, $z \in \{z_\ell, z_h\}$, follow a

[32] Alternatively, one can think of dynasties where offspring inherit the totality of the previous generation's wealth. This is similar to the formulation in Benhabib, Bisin and Zhu (2011). This model does not admit a stationary wealth distribution but remains tractable by focusing on the behavior of aggregates and wealth shares across entrepreneurial types.

Markov process with transition matrix

$$\mathbb{P} = \left[\begin{array}{cc} p & 1-p \\ 1-p & p \end{array} \right], \tag{77}$$

where $p \in (0,1)$ is the probability that an entrepreneur retains their productivity across periods. The autocorrelation coefficient of productivity is $\rho \equiv 2p-1$, so that productivity is persistent if $p > 1/2$ ($\rho > 0$). The symmetry in transition probabilities ensures that half of the entrepreneurs have high-productivity at any point in time, $\mu = 1/2$.

The dynamic problem of the entrepreneurs is now

$$V(a,z) = \max_{a'} \log\left(R(z)\,a - a'\right) + \beta \sum_{z'} \mathbb{P}(z' \mid z)\, V(a', z'), \tag{78}$$

where $R(z)$ is as in equation (9). The solution to this problem gives the same savings rule as before,

$$a' = \beta R(z)\, a. \tag{79}$$

This structure leads to equilibrium conditions paralleling those in Section 3. In particular, the neutrality result in Proposition 1 is preserved and productivity in the stationary equilibrium is endogenous and determined by a quadratic equation that is now[33]

$$0 = \left(1 - \rho\beta\left(1-\tau_a\right)\right) Z^2 - \left(1 + \rho\left(1 - 2\beta\left(1-\tau_a\right)\right)\right) \frac{z_h + z_\ell}{2} Z + \rho\left(1 - \beta\left(1-\tau_a\right)\right) z_h z_\ell. \tag{80}$$

Aggregate productivity in the stationary equilibrium depends now on ρ, the persistence of the entrepreneurial productivity process.

The main result out of this model is that the effects of wealth taxes on Z also depend on the persistence of productivity. We show that Z is increasing in the wealth tax if and only if entrepreneurial productivity is persistent, $\rho > 0$. As in Section 4, an increase in the wealth tax increases the returns of high-productivity entrepreneurs and reduces those of low-productivity entrepreneurs (see Lemma 10, Appendix E). This translates into a higher wealth share of high-productivity entrepreneurs (s_h) if and only if current H-type

[33] By studying this equation we prove that there exists a unique stationary equilibrium with heterogeneous returns, provided that the collateral constraint is not too lose, or, equivalently, that the wealth tax is sufficiently low. The analysis and derivations are essentially the same as the ones presented above and so we leave them to the Appendix.

entrepreneurs are expected to remain so in the future.[34]

Proposition 13. ***(Efficiency Gains from Wealth Taxation)*** *For all $\tau_a < \overline{\tau}_{a,\rho}$, an increase in the wealth tax (τ_a) increases aggregate productivity, $\frac{dZ}{d\tau_a} > 0$, if and only if entrepreneurial productivity is persistent, $\rho > 0$.*

The remainder of our results also have parallels in this model—which we omit for space considerations. Wealth taxes reduce average entrepreneurial returns and so entrepreneurs as a group see their welfare decrease when wealth taxes increase. Workers benefit through the increase in their income following the increase in productivity, as they did before. The choice of optimal taxes takes a familiar form, balancing the positive level effect on aggregate variables, with the decrease in returns.

9 Conclusions

In this paper, we have studied book-value wealth taxation and capital income taxation in an infinite horizon economy with heterogeneous entrepreneurial productivity. We showed an important neutrality result that distinguishes the two forms of taxation: capital income taxation has no effect on the steady state after-tax marginal product of capital and returns, whereas the wealth tax does. In particular, the wealth tax increases the dispersion of after-tax returns, thereby shifting aggregate wealth toward the high-productivity entrepreneurs and therefore raising aggregate productivity. In addition, when the government balances its budget, aggregate capital, output, and wages all rise.

We showed that the effects on welfare from a higher wealth tax differ across groups: workers and high-productivity entrepreneurs unambiguously benefit through higher wages and higher wealth growth, respectively, whereas low-productivity entrepreneurs typically lose. We then characterized the optimal combination of capital income and wealth taxes that balances these gains and losses and showed that it features a positive wealth tax if the increase in productivity that the wealth tax generates has a strong-enough pass-through into higher wages and capital, something that happens when the capital intensity in the economy, captured by the capital share α, is above a threshold (around 0.3 for a wide range of parameters).

[34]Our results on the interplay of the persistence of entrepreneurial productivity and wealth taxes in determining aggregate productivity extend those of Moll (2014). We show that wealth taxes reduce misallocation through their heterogeneous effect on returns (for a given degree of persistence) resulting in asset accumulation by high-productivity entrepreneurs in a similar way that higher persistence does.

We also studied how the form of capital taxation affects the distribution of entrepreneurial productivity through its effects on innovation and entrepreneurial effort. Raising the wealth tax increases innovation and entrepreneurial effort and, consequently, the equilibrium number of high-productivity entrepreneurs as well as their marginal product of capital. This, in turn, increases productivity, output, wages, and welfare. As a result, the optimal wealth tax is higher in the presence of endogenous innovation and entrepreneurial effort. At the core of these results are the powerful incentives provided by increasing return dispersion on individual choices.

One defining characteristic of our study is the focus on the taxation of the book-value of wealth rather than on its market-value. There are conceptual and practical reasons for this. First, taxes on the book-value of wealth operate very differently from taxes on the market-value because they do not tax current or future returns. We can see this in the context of our baseline model where the market-value of wealth of an entrepreneur with productivity z and a units of assets is given by the discounted value of the entrepreneur's assets (invested in risk-free bonds) and their future returns (that depend on their productivity):

$$\sum_{t=0}^{\infty}\left(\frac{1}{1+r}\right)^{t}\left((1+r)^{t}a+\delta^{t}\Pi\left(z,a_{t}\right)\right)=\underbrace{a}_{\text{Book-Value}}+\overbrace{\pi^{\star}\left(z\right)a}^{\text{Current Capital Income}}+\underbrace{\frac{\beta\delta^{2}\frac{R(z)}{1+r}}{1-\beta\delta^{2}\frac{R(z)}{1+r}}\pi^{\star}\left(z\right)a}_{\text{Future (unrealized) Capital Income}}. \tag{81}$$

A market-value tax is therefore a book-value of assets, plus a tax on current returns (profits), and a tax on future (unrealized) returns. So, conceptually, a tax on the market-value of wealth mixes the properties of book-value wealth taxes we have studied with those of a tax on (excess) returns, like the capital income tax.[35]

Second, the taxation of wealth at book value helps to address with many of the practical implementation issues raised by wealth taxes. While valuing the market value of infrequently-traded and closely-held assets is intrinsically hard, most tax agencies already have access to measures of the book value of private firms and other forms of wealth from standard accounting practices. This makes book-value wealth taxation a viable and theoretically grounded alternative to proposals of wealth taxation based on market values and to the more commonly used capital income tax based on realized returns.

[35]The returns of low-productivity entrepreneurs in our model are given by the rate of return on bonds, r. Profits capture excess returns above this rate (see equation 7) and are zero for these agents. Therefore, a book value wealth tax is the same as a market value wealth tax for individuals that have market returns.

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ONLINE APPENDIX

Not for Publication

A Further Equations for the Benchmark Model

A.1 Entrepreneur's Problem

Entrepreneurial Production. We start with an entrepreneur's labor demand choice given a level of capital:

$$\pi(z,k) = \max_n (zk)^{\alpha} n^{1-\alpha} - wn,$$

which yields the labor demand function in equation (5). Substituting (5) into (4) allows us to solve for the entrepreneur's capital choice in equation (6) that implies an optimal capital choice:

$$k^{\star}(z,a) = \begin{cases} \lambda a & \text{if } \alpha\left(\frac{1-\alpha}{w}\right)^{(1-\alpha)/\alpha} z > r \\ [0,\lambda a] & \text{if } \alpha\left(\frac{1-\alpha}{w}\right)^{(1-\alpha)/\alpha} z = r \\ 0 & \text{if } \alpha\left(\frac{1-\alpha}{w}\right)^{(1-\alpha)/\alpha} z < r. \end{cases}$$

Replacing the capital choice and (5) into (4) yields the optimal entrepreneurial income in (7).

Entrepreneurial Savings. Given constant taxes and prices, the savings problem is

$$V_i(a) = \max_{a'} \log\left(R_i a - a'\right) + \beta\delta V\left(a'\right),$$

where $R_i = R(z_i)$ is defined as in (9) for $i \in \{\ell, h\}$.

We solve the entrepreneur's saving problem via guess and verify. To this end, we guess that the value function of an entrepreneur with productivity z_i, $i \in \{\ell, h\}$, has the form

$$V_i(a) = m_i + n\log(a),$$

where $m_\ell, m_h, n \in \mathbb{R}$ are coefficients. Under this guess, the optimal savings choice is

$$a_i^{'} = \frac{\beta\delta n}{1+\beta\delta n} R_i a.$$

Replacing the savings rule into the value function gives

$$\begin{aligned} V_i(a) &= \log\left(R_i a - a_i^{'}\right) + \beta\delta V_i\left(a_i^{'}\right) \\ m_i + n\log(a) &= \log\left(R_i a - a_i^{'}\right) + \beta\delta m_i + \beta\delta n \log\left(a_i^{'}\right) \\ m_i + n\log(a) &= \beta\delta n\log(\beta\delta n) + (1+\beta\delta n)\log\left(\frac{R_i}{1+\beta\delta n}\right) + \beta\delta m_i + (1+\beta\delta n)\log(a) \end{aligned}$$

Matching coefficients we obtain

$$\begin{aligned} n &= 1 + \beta\delta n; \\ m_i &= \beta\delta n\log(\beta\delta n) + (1+\beta\delta n)\log\left(\frac{R_i}{1+\beta\delta n}\right) + \beta\delta m_i. \end{aligned}$$

The solution to the first equation is $n = \frac{1}{1-\beta\delta}$. This in turn delivers the optimal saving decision of the entrepreneur in equation (10) with constant saving rate $\beta\delta$.

Finally, we solve for the remaining coefficients from the system of linear equations:

$$m_i = \frac{1}{(1-\beta\delta)^2}\left[\log(\beta\delta)^{\beta\delta}(1-\beta\delta)^{1-\beta\delta} + \log R_i\right]$$

The value of an entrepreneur with productivity z_i, $i \in \{\ell, h\}$, is then

$$V_i(a) = \frac{\log(\beta\delta)^{\beta\delta}(1-\beta\delta)^{1-\beta\delta}}{(1-\beta\delta)^2} + \frac{1}{(1-\beta\delta)^2}\log R_i + \frac{1}{1-\beta\delta}\log(a).$$

A.2 Stationary Recursive Competitive Equilibrium

Definition. A stationary recursive competitive equilibrium, given a measure L of hand-to-mouth workers who inelastically supply labor, is an entrepreneurial value function V, entrepreneurial policy functions a' and c, entrepreneurial operating value and policy functions $\Pi^\star, k^\star$, and $n^\star$, prices r and w, and a distribution of entrepreneurs Γ such that

i. V satisfies the Bellman equation (8) for the entrepreneurs' consumption-saving problem, and a' and c are the corresponding policy functions, given prices r and w;

ii. $\Pi^\star$ is the solution to the entrepreneurs' production problem in (4), and $k^\star$ and $n^\star$ are the corresponding policy functions, given r and w;

iii. the labor markets clears: $L = \int n^\star(a,z)\,d\Gamma$;

iv. the capital (and bond) market clears: $\int k^\star(a,z)\,d\Gamma = \int a d\Gamma$;

v. the goods market clears:

$$wL + \int c(a,z)\,d\Gamma + \int a^{'}(a,z)\,d\Gamma = \int (zk^\star(a,z))^\alpha (n^\star(a,z))^{1-\alpha}\,d\Gamma;$$

vi. the distribution of wealth is constant over time and consistent with the saving choices of entrepreneurs, $a^{'}$, and the birth-death process.

B Proofs: Benchmark Model

This appendix presents the proofs for the results listed in the paper, covering Sections 2 to 5. We reproduce the statement of all results for the reader's convenience.

Lemma 1. ***(Aggregate Variables in Equilibrium)*** *In the stationary heterogeneous-return equilibrium defined in Proposition 7, aggregate output, the wage rate, the interest rate, and gross returns are given as follows:*

$$\begin{aligned} Y &= (ZK)^{\alpha} L^{1-\alpha} \\ w &= (1-\alpha) \left({}^{ZK}/{}_{L}\right)^{\alpha} \\ r &= \alpha \left({}^{ZK}/{}_{L}\right)^{\alpha-1} z_{\ell} \\ R_{\ell} &= (1-\tau_a) + (1-\tau_k)\, \alpha \left({}^{ZK}/{}_{L}\right)^{\alpha-1} z_{\ell} \\ R_{h} &= (1-\tau_a) + (1-\tau_k)\, \alpha \left({}^{ZK}/{}_{L}\right)^{\alpha-1} z_{\lambda}. \end{aligned}$$

Proof. We start by considering the labor market clearing condition

$$\mu n^{\star}(z_h, K_h) + (1-\mu)\, n^{\star}(z_{\ell}, K_{\ell}) = L.$$

Replacing for the optimal labor demand (5) we get

$$\begin{aligned} \left(\frac{1-\alpha}{w}\right)^{1/\alpha} (z_h \mu K_h + z_{\ell}(1-\mu) K_{\ell}) &= L; \\ \left(\frac{1-\alpha}{w}\right)^{1/\alpha} ZK &= L. \end{aligned}$$

Manipulating this expression we get wages as:

$$w = (1-\alpha) \left({}^{ZK}/{}_{L}\right)^{\alpha}.$$

In the stationary heterogeneous-return equilibrium the interest rate is given by the returns of the low-productivity entrepreneurs and so

$$r = \alpha \left(\frac{1-\alpha}{w}\right)^{\frac{1-\alpha}{\alpha}} z_{\ell} = \alpha \left({}^{ZK}/{}_{L}\right)^{\alpha-1} z_{\ell}$$

Then, the profit rate of the high-productivity entrepreneurs is (from 7)

$$\pi^{\star}(z_h) = \left(\alpha \left(\frac{1-\alpha}{w}\right)^{(1-\alpha)/\alpha} z_h - r\right) \lambda = \alpha \left({}^{ZK}/{}_{L}\right)^{\alpha-1} (z_h - z_{\ell})\, \lambda$$

and the gross returns of entrepreneurs are:

$$R_\ell = (1-\tau_a) + (1-\tau_k)\, r = (1-\tau_a) + (1-\tau_k)\,\alpha\,(ZK/L)^{\alpha-1}\, z_\ell$$

and

$$R_h = (1-\tau_a) + (1-\tau_k)\,(r+\pi^\star(z_h)) = (1-\tau_a) + (1-\tau_k)\,\alpha\,(ZK/L)^{\alpha-1}\, z_\lambda.$$

Finally, we consider aggregate output. We aggregate in terms of the aggregate capital of H- and L-type entrepreneurs because the ratio of labor to capital is constant across entrepreneurs, equation (5). The output of an individual entrepreneur with productivity z and capital k is:

$$y(z,k) = \left(\frac{1-\alpha}{w}\right)^{(1-\alpha)/\alpha} zk = (ZK/L)^{\alpha-1}\, zk,$$

where the second equality comes after replacing the equilibrium wage level. Aggregate output is the sum of the output produced by all the entrepreneurs:

$$Y = (ZK/L)^{\alpha-1}\,(z_h\mu K_h + z_\ell\,(1-\mu)\,K_\ell) = (ZK)^\alpha\, L^{1-\alpha}.$$

This completes the derivation of the results.

□

Proposition 1. ***(Capital Income Tax is Neutral for Returns. Wealth Tax is Not)*** *In the stationary heterogeneous-return equilibrium, the after-tax returns of the H-type and L-type are independent of the capital income tax rate but do depend on the wealth tax rate:*

$$R_\ell = 1-\tau_a + \left(\frac{1}{\beta\delta} - (1-\tau_a)\right)\frac{z_\ell}{Z} \quad \text{and} \quad R_h = 1-\tau_a + \left(\frac{1}{\beta\delta} - (1-\tau_a)\right)\frac{z_\lambda}{Z}.$$

In particular, the wealth tax has a "use-it-or-lose-it" effect that changes the dispersion of returns and therefore the level of wealth inequality, whereas the capital income tax has no distributional effects.

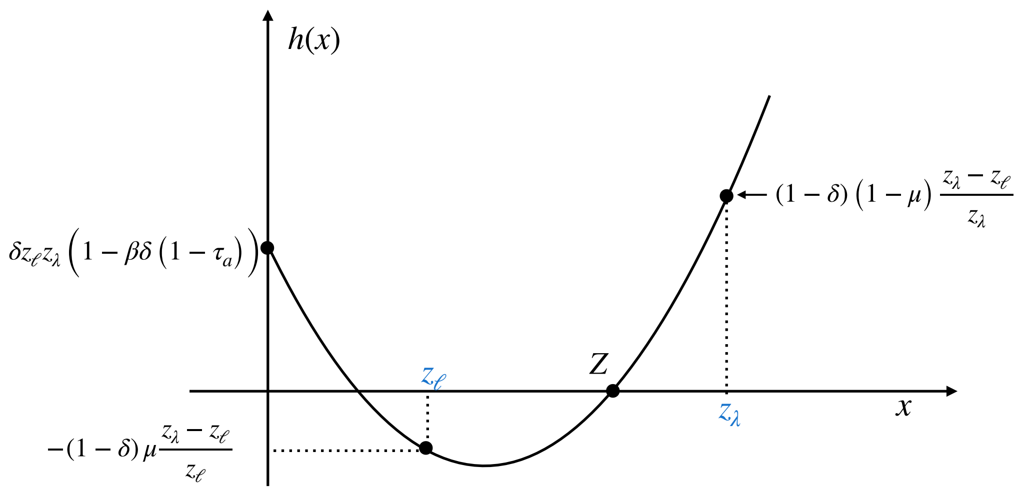
Proof. The proof is immediate by replacing (26) into the expression for returns of low- and high-productivity entrepreneurs in terms of aggregate variables obtained in Lemma 1. Moreover, the wealth-weighted return depends only on the entrepreneurial saving rate,

$$s_h R_h + (1-s_h)\,R_\ell = (1-\tau_a) + \left(\frac{1}{\beta\delta} - (1-\tau_a)\right)\frac{s_h z_\lambda + (1-s_h)\,z_\ell}{Z} = \frac{1}{\beta\delta}.$$

□

Proposition 2. ***(Existence and Uniqueness of Stationary "Heterogeneous-Return" Equilibrium)*** *A stationary competitive equilibrium exists and is unique if and only if λ satisfies Assumption 1. This equilibrium is characterized by an endogenous productivity level Z that*

Figure B.1: Stationary Competitive Equilibrium Productivity (Z)



Note: *The figure plots the polynomial* $h(x) = \left(1-\delta^2\beta\left(1-\tau_a\right)\right)x^2 - \left[(1-\delta)\left(\mu z_\lambda + (1-\mu)\,z_\ell\right) + \delta\left(1-\delta\beta\left(1-\tau_a\right)\right)\left(z_\lambda + z_\ell\right)\right]x + \delta\left(1-\delta\beta\left(1-\tau_a\right)\right)z_\ell z_\lambda = 0$ *that corresponds to equation (31). The stationary competitive equilibrium level of productivity corresponds to the larger root of h, marked with a circle on the horizontal axis.*

satisfies $z_\ell < Z < z_h$*, and features return heterogeneity* ($R_h > R_\ell$)*. In addition, the wealth share of the H-type satisfies* $s_h < 1/\lambda$*.*

Proof. We characterize the equilibrium level of productivity, Z, by studying the behavior of the quadratic equation in (31), depicted in Figure B.1. Specifically, we show that there is a single admissible root in the interval

$$Z \in \left(\max\left\{z_\ell, \frac{\delta\left(1-\eta\right)}{1-\delta\eta}z_\lambda\right\}, z_\lambda\right).$$

This interval is relevant for the proof of Lemma 2.

We start by defining the function

$$H\left(x\right) = \left(1-\delta\eta\right) - \frac{\left(1-\delta\right)\left(\mu z_\lambda + \left(1-\mu\right)z_\ell\right) + \delta\left(1-\eta\right)\left(z_\lambda + z_\ell\right)}{x} + \delta\left(1-\eta\right)\frac{z_\ell z_\lambda}{x^2}, \tag{82}$$

from the quadratic equation in (31), where $\eta \equiv \beta\delta\left(1-\tau_a\right)$. We verify directly that H has a root in the interval $\left(\max\left\{z_\ell, \frac{\delta(1-\eta)}{1-\delta\eta}z_\lambda\right\}, z_\lambda\right)$:

$$H\left(z_\ell\right) = -\frac{\left(1-\delta\right)\mu}{z_\ell}\left(z_\lambda - z_\ell\right) < 0$$
$$H\left(\frac{\delta\left(1-\eta\right)}{1-\delta\eta}z_\lambda\right) = -\frac{\left(1-\delta\eta\right)\left(1-\delta\right)\mu}{\delta\left(1-\eta\right)}\frac{1}{z_\lambda}\left(z_\lambda - z_\ell\right) < 0$$
$$H\left(z_\lambda\right) = \frac{\left(1-\delta\right)\left(1-\mu\right)}{z_\lambda}\left(z_\lambda - z_\ell\right) > 0$$

The existence of the unique root is guaranteed by the intermediate value theorem and the fact that the function is quadratic.

Now we derive necessary and sufficient conditions for the equilibrium productivity level to satisfy $Z \in (z_\ell, z_h)$, so that the equilibrium features heterogenous returns ($R_h > R_\ell$). Specifically we need $s_h < 1/\lambda$ to hold in equilibrium. This happens if and only if $Z < z_h$ (because $Z > z_\ell$ from its definition). So, we find a condition that guarantees that $H\left(z_h\right) > 0$ which implies that $Z < z_h$ because $H\left(Z\right) = 0$ and $H\left(z\right)$ is increasing in $z \geq Z$. The condition is

$$H\left(z_h\right) = \left(1-\delta\eta\right) - \frac{\left(1-\delta\right)\left(\mu z_\lambda + \left(1-\mu\right)z_\ell\right) + \delta\left(1-\eta\right)\left(z_\lambda + z_\ell\right)}{z_h} + \delta\left(1-\eta\right)\frac{z_\ell z_\lambda}{z_h^2} > 0,$$

after some manipulation this gives:

$$\lambda < \overline{\lambda} \equiv 1 + \frac{\left(1-\delta\right)\left(1-\mu\right)}{\left(1-\delta\right)\mu + \delta\left(1-\delta\beta\left(1-\tau_a\right)\right)\left(1-\frac{z_\ell}{z_h}\right)}.$$

Note that $\overline{\lambda} < {}^1\!/\!_\mu$ always and so it is the relevant bound for λ.

Finally, we verify that $z_h \geq \max\left\{z_\ell, \frac{\eta-\delta}{\eta}z_\lambda\right\}$. The first case is verified immediately, the second case applies if $\frac{\delta(1-\eta)}{1-\delta\eta} > \frac{z_\ell}{z_\lambda}$. A sufficient condition for $z_h \geq \frac{\delta(1-\eta)}{1-\delta\eta}z_\lambda$ is:

$$z_h \geq \frac{\delta\left(1-\eta\right)}{1-\delta\eta}z_\lambda$$
$$\left(1-\delta\right) \geq \delta\left(1-\eta\right)\left(\lambda-1\right)\left(1-\frac{z_\ell}{z_h}\right)$$
$$\left(1-\delta\right) \geq \delta\left(1-\eta\right)\left(\lambda-1\right)\left(1-\frac{\delta\left(1-\eta\right)}{1-\delta\eta}\right)$$
$$\frac{1-\delta\eta}{\delta\left(1-\eta\right)} \geq \lambda - 1$$

For this bound not to bind we need that it is above $\overline{\lambda}$:

$$\frac{1-\delta\eta}{\delta(1-\eta)} \geq \overline{\lambda} - 1$$

$$(1-\delta\eta)\left((1-\delta)\mu + \delta(1-\eta)\left(1-\frac{z_\ell}{z_h}\right)\right) \geq (1-\delta)\delta(1-\eta)(1-\mu)$$

$$(1-\delta\eta)\delta(1-\eta)\left(1-\frac{z_\ell}{z_h}\right) \geq (1-\delta)\left[\delta(1-\eta) - (\delta+1-2\delta\eta)\mu\right]$$

The condition is most stringent when $\mu = 0$ (counterfactually). This leads to a sufficient condition

$$(1-\delta\eta)\left(1-\frac{z_\ell}{z_h}\right) \geq (1-\delta)$$

$$\delta\frac{1-\eta}{1-\delta\eta} \geq \frac{z_\ell}{z_h}$$

which is verified by assumption. So the upper bound $\overline{\lambda}$ is sufficient for $z_h \geq \max\left\{z_\ell, \frac{\delta(1-\eta)}{1-\delta\eta}z_\lambda\right\}$.

□

Lemma 2. ***(Saving and Dissaving in the Stationary Equilibrium)*** *In the stationary heterogeneous-return equilibrium, the rates of return of the L-type and the H-type satisfy the following inequalities: $\beta\delta R_\ell < 1 < \beta\delta R_h < 1/\delta$. As a result, the wealth of the L-type (H-type) shrinks (grows) with age. Therefore, the H-type is wealthier than the L-type: $s_h > \mu$.*

Proof. We start by showing that $R_\ell < 1/\beta\delta < R_h$. We verify this directly using the expression for the returns of high- and low-productivity entrepreneurs, the fact that $z_\ell < Z < z_\lambda$, and the equilibrium condition for the return on capital:

$$R_\ell = (1-\tau_a) + (1-\tau_k)\alpha Z^\alpha\left(\frac{K}{L}\right)^{\alpha-1}\frac{z_\ell}{Z} < (1-\tau_a) + (1-\tau_k)\alpha Z^\alpha\left(\frac{K}{L}\right)^{\alpha-1} = \frac{1}{\beta\delta},$$

and

$$\frac{1}{\beta\delta} = (1-\tau_a) + (1-\tau_k)\alpha Z^\alpha\left(\frac{K}{L}\right)^{\alpha-1} < (1-\tau_a) + (1-\tau_k)\alpha Z^\alpha\left(\frac{K}{L}\right)^{\alpha-1}\frac{z_\lambda}{Z} = R_h,$$

Letting $\eta \equiv \delta\beta(1-\tau_a)$, we can also show that $\beta\delta R_h < 1/\delta$ if $\frac{\delta(1-\eta)}{1-\delta\eta}z_\lambda < Z$. Thus,

$$\beta\delta R_\ell < 1 < \beta\delta R_h < 1/\delta \longleftrightarrow Z \in \left(\max\left\{z_\ell, \frac{\delta(1-\eta)}{1-\delta\eta}z_\lambda\right\}, z_\lambda\right).$$

The interval for Z is non-empty. This is immediate because:

$$z_\ell < z_\lambda \qquad \text{and} \qquad \frac{\delta(1-\eta)}{1-\delta\eta} < 1.$$

Moreover, the lower bound depends on the ratio of productivities: $\max\left\{z_\ell, \frac{\delta(1-\eta)}{1-\delta\eta} z_\lambda\right\} = z_\ell$ if and only if $\frac{\delta(1-\eta)}{1-\delta\eta} \leq \frac{z_\ell}{z_\lambda}$. In the proof of Proposition 2 we establish that Z lies in the desired interval.

Finally, we prove that $s_h > \mu$. We know that $s_h = \frac{Z-z_\ell}{z_\lambda - z_\ell}$, so $s_h > \mu$ is equivalent to $Z > \mu z_\lambda + (1-\mu) z_\ell$. We can verify if this is the case by evaluating at $\mu z_\lambda + (1-\mu) z_\ell$ the residual of the quadratic equation H defined in (31):

$$H\left(\mu z_\lambda + (1-\mu) z_\ell\right) = -\delta (1-\eta)(1-\mu)\mu \left(\frac{z_\lambda - z_\ell}{\mu z_\lambda + (1-\mu) z_\ell}\right)^2 < 0$$

The residual is always negative. So it must be that $Z > \mu z_\lambda + (1-\mu) z_\ell$ and thus $s_h > \mu$.

□

Proposition 3. ***(Efficiency Gains from Wealth Taxation)*** *For all $\tau_a < \overline{\tau}_a$, a higher wealth tax increases the steady-state aggregate productivity level, $\frac{dZ}{d\tau_a} > 0$.*

Proof. We use the auxiliary function H defined in the proof of Proposition 2, equation (82). Simple manipulation of the function gives:

$$H(x;\tau_a) = F(x) - \left(1 - \frac{z_\ell}{x}\right)\left(1 - \frac{z_\lambda}{x}\right)\delta^2 \beta (1-\tau_a),$$

where $F(x)$ is a function of only x that does not depend on taxes. We now establish that H is decreasing in τ_a for $x \in (z_\ell, z_\lambda)$, which is the interval of the equilibrium value of Z:

$$\frac{d\tilde{H}(x,\tau_a)}{d\tau_a} = \underbrace{\left(1 - \frac{z_\ell}{x}\right)}_{(+)}\underbrace{\left(1 - \frac{z_\lambda}{x}\right)}_{(-)}\delta^2\beta < 0.$$

This implies that $\frac{dZ}{d\tau_a} > 0$ because, as we proved in proposition 2, H is increasing in x for $x \in (z_\ell, z_\lambda)$. See Figure 2 for a graphical version of this proof.

□

Proposition 4. ***(Wealth Tax Increases Equilibrium Dispersion of Returns)*** *For all $\tau_a < \overline{\tau}_a$, an increase in the wealth tax increases the rate of return of the H-type and reduces that of the L-type. That is,*

$$\frac{d\log R_h}{d\tau_a} \equiv \xi^{R_h}_{\tau_a} = \xi^{R_h}_{Z} \times \frac{d\log Z}{d\tau_a} > 0 \qquad \textit{and} \qquad \frac{d\log R_\ell}{d\tau_a} \equiv \xi^{R_\ell}_{\tau_a} = \xi^{R_\ell}_{Z} \times \frac{d\log Z}{d\tau_a} < 0,$$

where $\xi^{R_i}_Z = \frac{d\log R_i}{d\log Z}$. Furthermore, the population-weighted average of returns and log returns decline *with τ_a:*

$$\frac{d\left(\mu R_\ell + (1-\mu) R_h\right)}{d\tau_a} < 0;$$

$$\frac{d\left(\mu \log R_h + (1-\mu)\log R_\ell\right)}{d\tau_a} = \left(\mu \xi_Z^{R_h} + (1-\mu)\,\xi_Z^{R_\ell}\right)\frac{d\log Z}{d\tau_a} < 0.$$

Proof. From the stationary level of wealth of high-productivity entrepreneurs we know that:

$$R_h = \frac{1}{\beta\delta^2}\left(1 - \frac{(1-\delta)\,\mu}{s_h}\right) \quad \longrightarrow \quad \frac{dR_h}{dZ} = \frac{(1-\delta)\,\mu}{\beta\delta^2}\frac{1}{s_h^2}\frac{ds_h}{dZ} > 0$$

A similar calculation delivers:

$$R_\ell = \frac{1}{\beta\delta^2}\left(1 - \frac{(1-\delta)(1-\mu)}{(1-s_h)}\right) \quad \longrightarrow \quad \frac{dR_\ell}{dZ} = -\frac{(1-\delta)(1-\mu)}{\beta\delta^2}\frac{1}{(1-s_h)^2}\frac{ds_h}{dZ} < 0.$$

With this we get:

$$\frac{d\left(\mu R_h + (1-\mu) R_\ell\right)}{d\tau_a} = \frac{1-\delta}{\beta\delta^2}\left(\frac{\left(\mu(1-s_h) + (1-\mu)s_h\right)(\mu - s_h)}{s_h^2(1-s_h)^2}\right)\frac{ds_h}{d\tau_a} < 0.$$

The sign follows because $s_h > \mu$ as proven above.

Finally, we consider the weighted product of returns, that is also decreasing in wealth taxes.

$$\begin{aligned}\frac{dR_h^{\mu}R_\ell^{1-\mu}}{d\tau_a} &= (1-\mu)\,R_h^{\mu}R_\ell^{-\mu}\frac{dR_\ell}{d\tau_a} + \mu R_h^{\mu-1}R_\ell^{1-\mu}\frac{dR_h}{d\tau_a} \\ &< R_h^{\mu}R_\ell^{1-\mu}\frac{(1-\delta)}{\beta\delta^2}R_\ell\left[\frac{\left(\mu(1-s_h) + (1-\mu)s_h\right)(\mu - s_h)}{s_h^2(1-s_h)^2}\right]\frac{ds_h}{d\tau_a}\end{aligned}$$

The inequality follows because $s_h < \mu$. This result implies that the average elasticity is negative.

□

Lemma 3. *For all $\tau_a < \overline{\tau}_a$, under Assumption 3, the steady-state level of capital is*

$$K = \left(\alpha\frac{\beta\delta(1-\theta)}{1-\beta\delta}\right)^{\frac{1}{1-\alpha}} Z^{\frac{\alpha}{1-\alpha}} L,$$

and the long run elasticities of aggregate variables with respect to productivity are

$$\xi_Z^K = \xi_Z^Y = \xi_Z^w = \frac{\alpha}{1-\alpha},$$

where $\xi_Z^x \equiv \frac{d\log x}{d\log Z}$. *Moreover, the wealth level of each entrepreneurial type is*

$$A_h = \frac{1}{\mu}\frac{Z - z_\ell}{z_\lambda - z_\ell}K \qquad \frac{dA_h}{dZ} \propto Z^{\frac{2\alpha-1}{1-\alpha}}\left(Z - \alpha z_\ell\right) > 0$$

$$A_\ell = \frac{1}{1-\mu}\frac{z_\lambda - Z}{z_\lambda - z_\ell}K \qquad \frac{dA_\ell}{dZ} \propto Z^{\frac{2\alpha-1}{1-\alpha}}\left(\alpha z_\lambda - Z\right) \lesseqgtr 0,$$

if and only if $\alpha z_\lambda \lesseqgtr Z$.

Proof. Using Assumption 3 and equation (26) the equilibrium level of capital is

$$K = \left(\alpha\frac{\beta\delta\left(1-\theta\right)}{1-\beta\delta}\right)^{\frac{1}{1-\alpha}} Z^{\frac{\alpha}{1-\alpha}}L,$$

which is increasing in Z. From this, it is immediate that $Y = \left(ZK\right)^{\alpha}L^{1-\alpha}$ is also increasing in Z. Replacing these results in equilibrium wages (Lemma 1) we get

$$w = \left(1-\alpha\right)\left({}^{ZK}\!/_{L}\right)^{\alpha} = \left(1-\alpha\right)\frac{Y}{L} = \left(1-\alpha\right)\left(\alpha\frac{\beta\delta\left(1-\theta\right)}{1-\beta\delta}\right)^{\frac{\alpha}{1-\alpha}} Z^{\frac{\alpha}{1-\alpha}}.$$

The elasticities follow immediately.

Because K and s_h increase with productivity, it must be the case that $A_h = \frac{s_h K}{\mu}$ increases as well. We are left with the response of A_ℓ. To get it we first write A_ℓ in terms of Z using the definition of the wealth share of the high-productivity entrepreneurs:

$$A_\ell = \frac{\left(1-s_h\right)K}{1-\mu} = \frac{1}{1-\mu}\left(1 - \frac{Z - z_\ell}{z_\lambda - z_\ell}\right)K = \frac{1}{1-\mu}\left(\alpha\frac{\beta\delta\left(1-\theta\right)}{1-\beta\delta}\right)^{\frac{1}{1-\alpha}} L\frac{z_\lambda - Z}{z_\lambda - z_\ell}Z^{\frac{\alpha}{1-\alpha}}$$

Taking derivatives shows that A_ℓ decreases with Z (and hence with τ_a):

$$\frac{dA_\ell}{dZ} \propto \frac{Z^{\frac{\alpha}{1-\alpha}-1}}{z_\lambda - z_\ell}\left[\alpha z_\lambda - Z\right]$$

which is negative if $\alpha z_\lambda < Z$.

□

Lemma 4. *Assume that total government is fixed,* $G + T = \overline{\theta}$. *Then, the (semi-)elasticities of capital, output, and wages to a change in the wealth tax satisfy*

$$\xi_{\tau_a}^K = \xi_{\tau_a}^Y = \xi_{\tau_a}^w > \frac{\alpha}{1-\alpha}\frac{d\log Z}{d\tau_a}.$$

Proof. The proof is immediate and follows from the fact that the increase in Y under Lemma 3 also increases the revenue raised. Holding revenue constant allows for a larger decrease in capital income taxes in response to wealth taxes.

□

Proposition 5. *For all $\tau_a < \overline{\tau}_a$, under Assumption 3, an increase in the wealth tax increases the welfare of workers and newborn H-type entrepreneurs,*

$$\frac{dV_w}{d\tau_a} > 0 \qquad and \qquad \frac{dV_h(\overline{a})}{d\tau_a} > 0.$$

Moreover, an increase in the wealth tax increases the welfare of newborn L-type entrepreneurs and the ex ante welfare of newborn entrepreneurs under the following conditions:

$$\frac{dV_\ell(\overline{a})}{d\tau_a} > 0 \quad if \quad \xi_Z^K > \frac{-1}{1-\beta\delta}\xi_Z^{R_\ell};$$
$$\frac{d\left(\mu V_h(\overline{a}) + (1-\mu)V_\ell(\overline{a})\right)}{d\tau_a} > 0 \quad if \quad \xi_Z^K > \frac{-1}{1-\beta\delta}\left(\mu\xi_Z^{R_h} + (1-\mu)\xi_Z^{R_\ell}\right).$$

Proof. We begin with worker welfare. Recall that $V_w = \frac{1}{1-\beta\delta}\log(w+T)$ and so $dV_w/d\tau_a = \frac{1}{1-\beta\delta}\xi_{w+T} d\log Z/d\tau_a$, where $\xi_{w+T} = \alpha/1-\alpha > 0$ is the elasticity of worker income with respect to productivity. Recall that $w + T = ((1-\alpha) + \theta_T\alpha)Y/L$ under Assumption (3), and so the elasticity with respect to productivity is $\xi_{w+T}^Z \equiv d\log(w+T)/dZ = \frac{\alpha}{1-\alpha}$. Finally, $d\log Z/d\tau_a > 0$ from proposition 3. This gives the result.

The value of a newborn entrepreneur with productivity z_i, for $i \in \{\ell, h\}$, is $V_i(\overline{a}) = m_i + \frac{1}{1-\beta\delta}\log(\overline{a})$, where $m_i = \frac{1}{(1-\beta\delta)^2}[\beta\delta\log\beta\delta + (1-\beta\delta)\log(1-\beta\delta) + \log R_i]$. Recall that $\overline{a} = K$. Hence, the change in the welfare of that entrepreneur when the wealth tax increases is

$$\frac{dV_i(\overline{a})}{d\tau_a} = \frac{1}{1-\beta\delta}\left(\xi_K^Z + \frac{1}{1-\beta\delta}\xi_{R_i}^Z\right)\frac{d\log Z}{d\tau_a}.$$

It is immediate that $dV_h(\overline{a})/d\tau_a > 0$ because $\xi_K^Z, \xi_{R_h}^Z > 0$ from Proposition 4 and Lemma 3.

For L-type entrepreneurs to benefit from an increase in the wealth tax it must be that

$$\xi_K^Z > \frac{-1}{1-\beta\delta}\xi_{R_\ell}^Z.$$

Finally, a newborn entrepreneur benefits from an increase in the wealth tax it must be that

$$\xi_K^Z > \frac{-1}{1-\beta\delta}\left(\mu\xi_{R_h}^Z + (1-\mu)\xi_{R_\ell}^Z\right).$$

Recall that $\xi_K^Z = \alpha/1-\alpha$ and so there are cutoffs on the wealth tax above which L-type and newborn entrepreneurs benefit.

□

Proposition 6. ***(Optimal Taxes)*** *Under Assumption 3, there exists a unique tax combination* $(\tau_a^\star, \tau_k^\star)$ *that maximizes the utilitarian welfare. An interior solution* $\tau_a^\star < \overline{\tau}_a$ *is the solution to:*

$$0 = \left[n_w \xi_Z^{w+T} + (1 - n_w)\, \xi_Z^K + \frac{1 - n_w}{1 - \beta\delta} \left(\mu \xi_Z^{R_h}(\tau_a) + (1 - \mu)\, \xi_Z^{R_\ell}(\tau_a)\right)\right] \frac{d \log Z}{d\tau_a}$$

Furthermore, there are two cutoff values for α, $\underline{\alpha}$ *and* $\overline{\alpha}$, *such that* $(\tau_a^\star, \tau_k^\star)$ *has the following properties:*

$$\begin{aligned} &\tau_a^\star \in \left[1 - \frac{1}{\beta\delta}, 0\right) \text{ and } \tau_k^\star > \theta && \text{if } \alpha < \underline{\alpha} \\ &\tau_a^\star \in \left[0, \frac{\theta(1-\beta\delta)}{\beta\delta(1-\theta)}\right] \text{ and } \tau_k^\star \in [0, \theta] && \text{if } \underline{\alpha} \le \alpha \le \bar{\alpha} \\ &\tau_a^\star \in \left(\frac{\theta(1-\beta\delta)}{\beta\delta(1-\theta)}, \tau_a^{\max}\right) \text{ and } \tau_k^\star < 0 && \text{if } \alpha > \bar{\alpha} \end{aligned}$$

where $\tau_a^{\max} \geq 1$, $\underline{\alpha}$ *and* $\overline{\alpha}$ *are the solutions to equation (59) with* $\tau_a = 0$ *and* $\tau_a = \frac{\theta(1-\beta\delta)}{\beta\delta(1-\theta)}$, *respectively. When* $\theta = 0$ *and there are no revenue needs, so* $\underline{\alpha} = \overline{\alpha}$.

Proof. The first order condition of the government's problem in (57) is

$$\begin{aligned} 0 &= n_w \frac{d \log w + T}{d\tau_a} + (1 - n_w) \frac{d \log \overline{a}}{d\tau_a} + \frac{1 - n_w}{1 - \beta\delta} \left(\mu \frac{d \log R_h}{d\tau_a} + (1 - \mu) \frac{d \log R_\ell}{d\tau_a}\right), \\ 0 &= \left[n_w \frac{d \log w + T}{d \log Z} + (1 - n_w) \frac{d \log K}{d \log Z} + \frac{1 - n_w}{1 - \beta\delta} \left(\mu \frac{d \log R_h}{d \log Z} + (1 - \mu) \frac{d \log R_\ell}{d \log Z}\right)\right] \frac{d \log Z}{d\tau_a}, \\ 0 &= \left[n_w \xi_{w+T}^Z + (1 - n_w)\, \xi_K^Z + \frac{1 - n_w}{1 - \beta\delta} \left(\mu \xi_{R_h}^Z + (1 - \mu)\, \xi_{R_\ell}^Z\right)\right] \frac{d \log Z}{d\tau_a}. \end{aligned}$$

From proposition 3 we know that $d \log Z / d\tau_a > 0$, so that an interior solution must equate the first term to zero.

We know that $\xi_{w+T}^Z = \xi_K^Z = \alpha/1-\alpha$ from Lemma 3 and that $\mu \xi_{R_h}^Z + (1 - \mu)\, \xi_{R_\ell}^Z < 0$ from Proposition 4. Further, the elasticities of returns are independent of α. Because of this we can define cutoffs for α by evaluating the right hand side of the equation at $\tau_a = 0$ and $\tau_a = \frac{\theta(1-\beta\delta)}{\beta\delta(1-\theta)}$. If α is exactly equal to the cutoff then the optimal τ_a is either 0 or $\frac{\theta(1-\beta)}{\beta(1-\theta)}$. The monotonicity of the right hand side lets us define the intervals shown in the proposition and the uniqueness of the solution. To see the monotonicity consider the formulas for $\xi_{R_h}^Z$ and $\xi_{R_\ell}^Z$ Proposition 4.

□

C Innovation Effort

To show that an equilibrium exists, we establish the existence of a unique fixed point on innovation effort (equivalently on the share of high-productivity entrepreneurs), where effort implies productivity that implies in turn the original level of effort. This is captured by a mapping $\varphi : \mathcal{M} \rightarrow \mathcal{M}$ that takes as an input a share of high-productivity entrepreneurs, $\mu \in \mathcal{M}$, and provides the implied level of effort; hence, $\varphi(\mu) \equiv e^{\star}(Z(\mu)) \in \mathcal{M}$. The existence of the fixed point for φ follows from standard fixed point arguments relying on Cellina's and Brouwer's fixed point theorems (Border, 1985, Thms. 15.1, 16.1).

Uniqueness of the equilibrium follows from the monotonicity of the equilibrium mapping φ and standard comparative statics results for fixed points. To see this, we first describe the mapping $Z(\mu)$ from μ to equilibrium Z and then how Z affects innovation effort in $e^{\star}(Z)$.

We state a series of intermediate lemmas and then join them to prove our main result.

We start by inspecting equation (31) and show that equilibrium productivity is increasing in the share of high-productivity entrepreneurs.

Lemma 5. *The equilibrium level of productivity, $Z(\mu)$, is increasing in the share of high-productivity entrepreneurs, μ.*

Proof. Define the correspondence $\gamma(\mu) \equiv \min\left\{z_h, \text{Roots}^{+}(H, \mu)\right\}$ as the largest admissible root of the quadratic function H, as defined in (31), that determines equilibrium productivity. We want to show that the function $\gamma(\mu)$ is increasing in μ. An increase in μ increases the magnitude of the linear term in . Because the linear term is always negative, the increase in magnitude increases the value of the highest root of H. This proves the result.

□

Then, from Proposition 1 we can see that steady-state returns are decreasing in Z, in such a way that return dispersion declines with productivity given a wealth tax rate τ_a. As a result, innovation effort declines in Z.

Lemma 6. *Innovation effort, $e^{\star}(Z)$, is decreasing in the level of productivity Z.*

Proof. Define the function $f(\mu, Z) \equiv \max\left\{\min\left\{\left(\Lambda'\right)^{-1}(\log R_h - \log R_\ell), 1\right\}, 0\right\}$ as the solution to (62), where $\left(\Lambda'\right)^{-1}$ is the inverse of the derivative of Λ. We want to show that the function $f(\mu, Z)$ is decreasing in Z (we already know it is independent of μ). To get the result, we show that an increase in Z decreases the dispersion in (log) returns, $\frac{d}{dZ}(\log R_h - \log R_\ell) < 0$. We show this directly using the expression of equilibrium returns as a function of Z in equation (28),

$$\frac{d}{dZ}\log R_\ell = \frac{\frac{d}{dZ}\left(1+\left(\frac{1}{\beta\delta(1-\tau_a)}-1\right)\frac{z_\ell}{Z}\right)}{1+\left(\frac{1}{\beta\delta(1-\tau_a)}-1\right)\frac{z_\ell}{Z}} = -\frac{\left(\frac{1}{\beta\delta(1-\tau_a)}-1\right)\frac{z_\ell}{Z^2}}{1+\left(\frac{1}{\beta\delta(1-\tau_a)}-1\right)\frac{z_\ell}{Z}};$$

$$\frac{d}{dZ}\log R_h = \frac{\frac{d}{dZ}\left(1+\left(\frac{1}{\beta\delta(1-\tau_a)}-1\right)\frac{z_\lambda}{Z}\right)}{1+\left(\frac{1}{\beta\delta(1-\tau_a)}-1\right)\frac{z_\lambda}{Z}} = -\frac{\left(\frac{1}{\beta\delta(1-\tau_a)}-1\right)\frac{z_\lambda}{Z}}{1+\left(\frac{1}{\beta\delta(1-\tau_a)}-1\right)\frac{z_\lambda}{Z}}.$$

Joining

$$\frac{d\left(\log R_h - \log R_\ell\right)}{dZ} = \frac{\left(\frac{1}{\beta\delta(1-\tau_a)}-1\right)\frac{z_\ell}{Z^2}}{1+\left(\frac{1}{\beta\delta(1-\tau_a)}-1\right)\frac{z_\ell}{Z}} - \frac{\left(\frac{1}{\beta\delta(1-\tau_a)}-1\right)\frac{z_\lambda}{Z}}{1+\left(\frac{1}{\beta\delta(1-\tau_a)}-1\right)\frac{z_\lambda}{Z}} < \frac{\left(\frac{1}{\beta\delta(1-\tau_a)}-1\right)\frac{(z_\ell-z_\lambda)}{Z^2}}{1+\left(\frac{1}{\beta\delta(1-\tau_a)}-1\right)\frac{z_\lambda}{Z}} < 0.$$

The decrease in the dispersion of (log) returns implies lower effort from the solution to (62).

□

Remark. These results describe the mapping from an arbitrary level of Z to returns and innovation. This is the relevant mapping for constructing the fixed point that constitutes an equilibrium, when τ_a and all the model's parameters are held fixed. It is this mapping from productivity to return dispersion that is decreasing in productivity. This is different from the result established in Proposition 4 that takes into account the equilibrium conditions of the economy (that is, taking into account that Z and s_h adjust to satisfy equation 31 when τ_a changes).

Proposition 7. ***(Existence of a Unique Stationary Equilibrium with Innovation)*** *There exists an upper bound for the wealth tax $\overline{\tau}_a^\mu$ such that for $\tau_a < \overline{\tau}_a^\mu$ there is a unique stationary equilibrium that features heterogeneous returns. That is, there is a unique level of the share of H-type entrepreneurs, $\mu^\star$, such that the optimal level of effort exerted by innovators satisfies $\mu^\star = e^\star\left(Z\left(\mu^\star\right)\right)$, and $Z\left(\mu^\star\right) \in \left(z_\ell, z_h\right)$ satisfies equation (31). The upper bound for the wealth tax satisfies*

$$\overline{\tau}_a^\mu = 1 - \frac{1}{\beta\delta}\left(1 - \frac{1-\delta}{\delta}\frac{1-\lambda\mu^\star\left(\overline{\tau}_a^\mu\right)}{(\lambda-1)\left(1-\frac{z_\ell}{z_h}\right)}\right),$$

where we make the dependence of $\mu^\star$ on τ_a explicit.

Proof. We first tackle the existence and then the uniqueness of the equilibrium.

Existence We provide two proofs of this result. The first one is longer but proves to be instructive of the workings of the model. It relies on Cellina's fixed point theorem, as found in Border (1985). The second one is more direct and relies on Brouwer's fixed point theorem. The objective in both cases is to show that the mapping of the share of high-productivity entrepreneurs into itself, defined by (62) (and the other equilibrium conditions), has a fixed point in the space $\mathcal{M} \equiv [0, 1]$.

We start by stating Cellina's fixed point theorem. The theorem breaks the construction of a mapping $\varphi : \mathcal{M} \to \mathcal{M}$ in two steps that capture how the share of high-productivity entrepreneurs μ implies a level of productivity Z that in turn implies a share μ through the level of returns. The theorem is as follows:

Theorem. ***[Cellina 1969; Border 1985, Thm. 15.1]*** *Let $\mathcal{M} \subseteq \mathbb{R}^m$ be nonempty, compact, and convex. Let $\varphi : \mathcal{M} \rightrightarrows \mathcal{M}$ be a correspondence defined on K. Suppose there is a nonempty-, compact-, and convex-valued correspondence $\gamma : \mathcal{M} \rightrightarrows \mathcal{K}$ defined on $\mathcal{M}$ with values in $\mathcal{K} \subseteq \mathbb{R}^n$, a compact and convex set, and also a continuous function $f : \mathcal{M} \times \mathcal{K} \to \mathcal{M}$ such that, for every $\mu \in \mathcal{M}$, $\varphi(\mu) = \{f(\mu, Z) \,|\, Z \in \gamma(\mu)\}$. Then, φ has a fixed point.*

To apply Cellina's theorem we set $\mathcal{M} \equiv [0,1]$ as the space of shares, with typical element μ, and $\mathcal{K} = [z_\ell, z_h]$ as the space of productivities with typical element Z. Both sets are nonempty, compact and convex, satisfying the theorem's requirements.

We then define the correspondence $\gamma(\mu) \equiv \min\left\{z_h, \text{Roots}^+(H, \mu)\right\}$ as the largest admissible root of the quadratic function H, defined in (31). This correspondence determines the equilibrium productivity. From Proposition 2 we know that γ is a function (a single-valued correspondence), and hence γ is nonempty-, compact-, and convex-valued.

Next, we define the function $f(\mu, Z) \equiv \max\left\{\min\left\{\left(\Lambda'\right)^{-1}(\log R_h - \log R_\ell), 1\right\}, 0\right\}$ as the solution to (62), where $\left(\Lambda'\right)^{-1}$ is the inverse of the derivative of Λ. This inverse exists and is continuous because Λ is convex and twice-continuously-differentiable. f takes as given μ and Z and provides a value of optimal effort, that gives a new value of μ. Notice that μ does not enter directly into f because returns are entirely determined given Z and τ_a, as seen in equation (28). So, the function is immediately (and vacuously) continuous in μ. The returns are themselves continuous in Z, see (28), so that f is continuous in Z.

Finally, we define the correspondence as $\varphi(\mu) \equiv \{f(\mu, Z) \,|\, Z = \gamma(\mu)\}$. All the conditions are satisfied and therefore a fixed point of φ exists. Any such fixed point is an equilibrium level for effort and the share of high-productivity entrenrepreneus of the economy, $\mu^\star$. This level of μ in turn implies the equilibrium level of productivity and other aggregate variables.

We now provide an alternative, and more direct proof based on Brouwer's fixed point theorem.

Theorem. ***[Brouwer 1912; Border 1985, Thm. 6.1]*** *Let $\mathcal{M} \subseteq \mathbb{R}^m$ be nonempty, compact, and convex. Let $\varphi : \mathcal{M} \to \mathcal{M}$ be a continuous function defined on K. Then, φ has a fixed point.*

To apply Brouwer's theorem we need to show that the function

$$\varphi(\mu) \equiv f(\mu, \gamma(\mu)) = f\left(\mu, \min\left\{z_h, \text{Roots}^+(H, \mu)\right\}\right)$$

is continuous, with f and γ defined as above. This in fact the case because the roots of the quadratic equation H, the minimum, the maximum, and f are all continuous.

Uniqueness This follows from showing that innovation effort, $\varphi(\mu) = e^{\star}(Z(\mu))$, is decreasing in the share of H-type entrepreneurs, μ. The result is immediate from combining Lemmas 5 and 6 because Z is increasing in μ, and e is decreasing in Z. Therefore, $\varphi(\mu)$ is monotonically decreasing, implying that there can be at most one fixed point in $[0, 1]$.

Condition on wealth taxes For the equilibrium to exhibit heterogeneous returns it must be that $\tau_a < \overline{\tau}_a$ (Assumption 2). However, $\overline{\tau}_a$ depends on μ, which now responds endogenously to τ_a. This means that equation (35) defines the upper bound on τ_a, given by $\overline{\tau}_a^{\mu}$, implicitly, something we emphasize by writing $\mu^{\star}(\overline{\tau}_a^{\mu})$ as the equilibrium level of μ when the wealth tax is $\overline{\tau}_a^{\mu}$.

□

Proposition 8. ***(Innovation Gains from Wealth Taxation)*** *For all $\tau_a < \overline{\tau}_a^{\mu}$, an increase in the wealth tax (τ_a) increases the equilibrium share of high-productivity entrepreneurs, $\mu^{\star}$. Capital income taxes do not affect innovation.*

Proof. The proof uses Theorem 3 in Villas-Boas (1997):

Theorem. ***[Villas-Boas 1997, Thm. 3]*** *Consider the mapping $\varphi_1 : \mathcal{M} \to \mathcal{M}$, the mapping $\varphi_2 : \mathcal{M} \to \mathcal{M}$, and a transitive, and reflexive order order $\geq$ on the set $\mathcal{M}$, such that both φ_1 and φ_2 have at least one fixed point in $\mathcal{M}$. If*

i. φ_1 is a weakly decreasing mapping, i.e., $\forall_{\mu',\mu\in\mathcal{M}}\, \mu' \geq \mu \longrightarrow \varphi_1\left(\mu'\right) \leq \varphi_1(\mu)$;

ii. φ_1 is higher than φ_2, that is $\varphi_1(\mu) > \varphi_2(\mu)$ for all $\mu \in \mathcal{M}$,

then, there is no fixed point $\mu_2^{\star}$ of φ_2 which is $>$ than a fixed point $\mu_1^{\star}$ of φ_1.

Remark. The theorem can be strengthened as it implies that the two mappings cannot have the same interior fixed point, so that we can conclude that for any (interior) fixed point $\mu_2^{\star}$ of φ_2 and any (interior) fixed point $\mu_1^{\star}$ of φ_1, it holds that $\mu_1^{\star} > \mu_2^{\star}$. To see this, consider a fixed point $\mu_2^{\star}$ of φ_2 a fixed point $\mu_1^{\star}$ of φ_1. We already know that $\mu_1^{\star} \geq \mu_2^{\star}$ from the Theorem. Now, suppose that $\mu_2^{\star} = \mu_1^{\star} = \mu^{\star}$ and that $\mu^{\star}$ is interior. Because φ_1 is higher than φ_2 and $\mu^{\star}$ is a common fixed point we have $\mu^{\star} = \varphi_1(\mu^{\star}) > \varphi_2(\mu^{\star}) = \mu^{\star}$, which is a contradiction.

We now turn to verify the conditions of the Theorem. Our space of interest is $\mathcal{M} \equiv [0, 1]$, and so we take the order $\geq$ to be the natural order on $\mathbb{R}$, which is transitive and reflexive. We define the mappings as $\varphi_1(\mu) \equiv \varphi\left(\mu, \tau_a^1\right)$ and $\varphi_2(\mu) \equiv \varphi\left(\mu, \tau_a^2\right)$ with $\overline{\tau}_a^{\mu} > \tau_a^1 > \tau_a^2$ and φ as in Proposition 7. These mappings have each a unique fixed point in $K \equiv [0, 1]$.

We know that that φ is decreasing from Lemmas 5 and 6 and so φ_1 satisfies the first condition.

To verify the second condition of the theorem, we establish that an increase in the wealth tax increases effort for any given level of the share of high-productivity entrepreneurs. Crucially, this condition speaks to the behavior of φ for any fixed level of μ as τ_a changes. Thus, the setup of Section 4 applies. In particular, Propositions 3 and 4, which shows that $\frac{dZ}{d\tau_a} > 0$, $\frac{dR_h}{dZ} > 0$ and

$\frac{dR_\ell}{dZ} < 0$, imply that the dispersion of returns increases with τ_a when holding μ fixed, that is, $d(\log R_h - \log R_\ell)/d\tau_a > 0$. This leads to a higher level of effort from equation (62).

All the conditions for the theorem are verified and so it must be that all the (interior) equilibrium shares of high-productivity entrepreneurs under the higher wealth tax, τ_a^1, are higher as the equilibrium shares under the low wealth tax, τ_a^2.

Remark. In establishing the second condition for the theorem we make use of Proposition 4 instead of Lemma 6. The difference lies in the nature of the mapping being constructed. The mapping required for the construction of φ in this proof takes into account the equilibrium response of Z to μ and to τ_a, while the one constructed in Lemma 6 captures how arbitrary levels of productivity affect returns, and, through them, the innovation effort, holding τ_a fixed.

□

Proposition 9. ***(Productivity Gains from Wealth Taxation with Innovation)*** *For all $\tau_a < \overline{\tau}_a^{\mu}$, an increase in the wealth tax (τ_a) increases productivity, $Z^\star$.*

Proof. This result follows from Propositions 3 and 8. Proposition 3 establishes that Z is increasing in τ_a holding μ fixed, and Lemma 5 establishes that Z is increasing in μ. Proposition 8 establishes that μ is increasing in τ_a for $\tau_a < \overline{\tau}_a^{\mu}$. Together they imply the result.

□

Proposition 10. *Under Assumption 3, an interior solution ($\tau_{a,\mu}^\star < \overline{\tau}_a^{\mu}$) to the optimal tax combination $\left(\tau_{a,\mu}^\star, \tau_{k,\mu}^\star\right)$ that maximizes the newborn welfare, $\mathcal{W}$, is the solution to the following equation:*

$$0 = \left(n_w \xi_Z^{w+T} + (1-n_w)\,\xi_Z^K + \frac{1-n_w}{1-\beta\delta}\left(\mu\xi_Z^{R_h} + (1-\mu)\,\xi_Z^{R_\ell}\right)\right)\frac{d\log Z}{d\tau_a} + \frac{1-n_w}{1-\beta\delta}\left(\mu\xi_\mu^{R_h} + (1-\mu)\,\xi_\mu^{R_\ell}\right)\frac{d\mu}{d\tau_a}$$

where $\xi_Z^x \equiv \frac{\partial \log x}{\partial \log Z}$ is the elasticity of variable x with respect to Z and $\xi_\mu^x \equiv \frac{\partial \log x}{\partial \mu}$ is the (semi-)elasticity with respect to μ. Recall from Lemma 3 that $\xi_Z^{w+T} = \xi_Z^K = \frac{\alpha}{1-\alpha}$.

Proof. The government's problem is still given by 57 but newborn welfare now includes the cost of innovation effort. An interior solution satisfies the first order condition

$$0 = \frac{d\mathcal{W}}{d\tau_a} = n_w \frac{dV_w}{d\tau_a} + (1-n_w)\frac{d}{d\tau_a}\left(\mu V_h(\overline{a}) + (1-\mu)\,V_\ell(\overline{a}) - \frac{\Lambda(\mu)}{(1-\beta\delta)^2}\right).$$

Replacing for the values of workers and entrepreneurs as in Section 5,

$$0 = n_w \frac{d \log w + T}{d\tau_a} + (1 - n_w) \frac{d \log (\overline{a})}{d\tau_a} + \frac{1 - n_w}{1 - \beta\delta} \left(\frac{d (\mu \log R_h + (1 - \mu) \log R_\ell)}{d\tau_a} - \frac{d\Lambda (\mu)}{d\tau_a} \right)$$

$$\begin{aligned} 0 = n_w \frac{d \log w + T}{d\tau_a} &+ (1 - n_w) \frac{d \log (\overline{a})}{d\tau_a} \\ &+ \frac{1 - n_w}{1 - \beta\delta} \left(\left(\mu \frac{d \log R_h}{d\tau_a} + (1 - \mu) \frac{d \log R_\ell}{d\tau_a} \right) + \left[\underbrace{(\log R_h - \log R_\ell) - \Lambda^{'} (\mu)}_{=0} \right] \frac{d\mu}{d\tau_a} \right) \end{aligned}$$

The last term is equal to zero because individuals already optimize over their innovation effort, so that, by the Pareto principle, taxes cannot improve on their choice. This leaves the local effects of the wealth tax taking μ as given (at its equilibrium level). We can further simplify these effects by noticing that, given μ, the effect of the wealth tax is only felt through the change in equilibrium productivity. This gives,

$$0 = \left(n_w \frac{d \log w + T}{d \log Z} + (1 - n_w) \frac{d \log (\overline{a})}{d \log Z} \right) \frac{d \log Z}{d\tau_a} + \frac{1 - n_w}{1 - \beta\delta} \left(\mu \frac{d \log R_h}{d\tau_a} + (1 - \mu) \frac{d \log R_\ell}{d\tau_a} \right).$$

We have left to obtain the change of returns taking into account the effect of taxes in innovation. We start from the representation of equilibrium returns in terms of μ and Z:

$$R_\ell = \frac{1}{\beta\delta^2} \left(1 - \frac{(1 - \delta) (1 - \mu) (z_\lambda - z_\ell)}{z_\lambda - Z} \right) \quad \text{and} \quad R_h = \frac{1}{\beta\delta^2} \left(1 - \frac{(1 - \delta) \mu (z_\lambda - z_\ell)}{Z - z_\ell} \right),$$

so that we can express the change in returns as

$$\frac{d \log R_i}{d\tau_a} = \frac{d \log R_i}{d \log Z} \frac{d \log Z}{d\tau_a} + \frac{d \log R_i}{d\mu} \frac{d\mu}{d\tau_a}.$$

Replacing in the first order condition we get the formula in the proposition.

We now turn to show that the sign of the average elasticity of returns with respect to productivity and innovation effort. We first obtaining a useful expression for the equilibrium level of μ. We know that (31) must be satisfied in equilibrium so we can write μ as follows,

$$\begin{aligned} \mu &= \frac{Z - z_\ell}{z_\lambda - z_\ell} \left(1 - \frac{\delta (1 - \eta)}{1 - \delta} \frac{(z_\lambda - Z)}{Z} \right) \\ 1 - \mu &= \frac{z_\lambda - Z}{z_\lambda - z_\ell} \left(1 + \frac{\delta (1 - \eta)}{1 - \delta} \frac{(Z - z_\ell)}{Z} \right) \end{aligned}$$

With this expression for μ we establish the effect of Z and μ on returns.

Average elasticity of returns with respect to productivity: From the expression for equilibrium returns we know that

$$\frac{\partial R_h}{\partial Z} = \frac{(1 - \delta) \mu}{\beta\delta^2} \frac{z_\lambda - z_\ell}{(Z - z_\ell)^2} \quad \text{and} \quad \frac{\partial R_\ell}{\partial Z} = - \frac{(1 - \delta) (1 - \mu)}{\beta\delta^2} \frac{z_\lambda - z_\ell}{(z_\lambda - Z)^2}.$$

We use this directly for the effect of productivity.

$$\begin{aligned}
\mu\frac{Z}{R_h}\frac{\partial R_h}{\partial Z}+(1-\mu)\frac{Z}{R_\ell}\frac{\partial R_\ell}{\partial Z} &= \frac{(1-\delta)(z_\lambda-z_\ell)}{\beta\delta^2}Z\left(\frac{1}{R_h}\left(\frac{\mu}{Z-z_\ell}\right)^2-\frac{1}{R_\ell}\left(\frac{1-\mu}{z_\lambda-Z}\right)^2\right)\\
&= (1-\delta)(z_\lambda-z_\ell)Z\left(\frac{\left(\frac{\mu}{Z-z_\ell}\right)^2}{\left(1-(1-\delta)(z_\lambda-z_\ell)\frac{\mu}{Z-z_\ell}\right)}\right.\\
&\qquad\left.-\frac{\left(\frac{1-\mu}{z_\lambda-Z}\right)^2}{\left(1-(1-\delta)(z_\lambda-z_\ell)\left(\frac{1-\mu}{z_\lambda-Z}\right)\right)}\right)\\
&= \frac{1}{(1-\delta)\delta(z_\lambda-z_\ell)}\left(\frac{((1-\delta\eta)Z-\delta(1-\eta)z_\lambda)^2}{\eta Z+(1-\eta)z_\lambda}\right.\\
&\qquad\left.-\frac{((1-\delta\eta)Z-\delta(1-\eta)z_\ell)^2}{\eta Z+(1-\eta)z_\ell}\right)\\
&< 0.
\end{aligned}$$

Average elasticity of returns with respect to innovation effort: We first get the derivative of returns with respect to μ:

$$\frac{\partial R_h}{\partial\mu}=-\frac{1}{\beta\delta^2}\frac{(1-\delta)(z_\lambda-z_\ell)}{Z-z_\ell}\qquad\text{and}\qquad\frac{\partial R_\ell}{\partial\mu}=\frac{1}{\beta\delta^2}\frac{(1-\delta)(z_\lambda-z_\ell)}{z_\lambda-Z}.$$

Now we use this to establish the effect of innovation on returns:

$$\begin{aligned}
\mu\frac{1}{R_h}\frac{\partial R_h}{\partial\mu}+(1-\mu)\frac{1}{R_\ell}\frac{\partial R_\ell}{\partial\mu} &= \frac{(1-\delta)}{\beta\delta^2}\left[-\frac{1}{R_h}\mu\frac{z_\lambda-z_\ell}{Z-z_\ell}+\frac{1}{R_\ell}(1-\mu)\frac{(z_\lambda-z_\ell)}{z_\lambda-Z}\right]\\
&= (1-\delta)\left[\frac{(1-\mu)\frac{(z_\lambda-z_\ell)}{z_\lambda-Z}}{1-(1-\delta)\frac{(1-\mu)(z_\lambda-z_\ell)}{z_\lambda-Z}}-\frac{\mu\frac{z_\lambda-z_\ell}{Z-z_\ell}}{1-(1-\delta)\frac{\mu(z_\lambda-z_\ell)}{Z-z_\ell}}\right]\\
&= \frac{1}{\delta}\left(\frac{(1-\delta\eta)Z-\delta(1-\eta)z_\ell}{\eta Z+(1-\eta)z_\ell}-\frac{(1-\delta\eta)Z-\delta(1-\eta)z_\lambda}{\eta Z+(1-\eta)z_\lambda}\right)\\
&> 0.
\end{aligned}$$

□

D Entrepreneurial Effort

Consider a model like that in Section 2 where entrepreneurs can exert effort every period to increase the level of productivity. We capture the effect of effort as modifying the production function of entrepreneurs to:

$$y = (zk)^{\alpha} g(e)^{\gamma} n^{1-\alpha-\gamma}.$$

where $\gamma \in [0,1)$. Exerting effort has a utility cost of $h(e)$, where $h^{'}(e) > 0$ and $h^{''}(e) \geq 0$. The utility function is now

$$u(c,e) = \log(c - h(e)).$$

D.1 Entrepreneurial Problem with Effort in Production

Entrepreneurial Production. We solve for an entrepreneur's static effort and labor demand choices. The solution is characterized by the following first order conditions:

$$u_e h^{'}(e) = (1-\tau_k) u_c \cdot \gamma (zk)^{\alpha} g(e)^{\gamma-1} n^{1-\alpha-\gamma} g^{'}(e); \qquad w = (1-\alpha-\gamma)(zk)^{\alpha} g(e)^{\gamma} n^{-\alpha-\gamma};$$

The second condition implies that

$$n = \left[\frac{(1-\alpha-\gamma)(zk)^{\alpha} g(e)^{\gamma}}{w}\right]^{\frac{1}{\alpha+\gamma}}.$$

Replacing back in the first order condition we obtain

$$\frac{u_e}{u_c}\frac{h^{'}(e)}{g^{'}(e)} = (1-\tau_k)\gamma (zk)^{\frac{\alpha}{\alpha+\gamma}} g(e)^{\frac{-\alpha}{\alpha+\gamma}} \left(\frac{1-\alpha-\gamma}{w}\right)^{\frac{1-\alpha-\gamma}{\alpha+\gamma}}.$$

For tractability we impose that $\frac{h^{'}(e)}{g^{'}(e)} = \psi$ is constant, say with $h(e) = \psi e$ and $g(e) = e$. Then the last condition gives the optimal effort and labor choices given prices, taxes, and the level of capital of the entrepreneur:

$$e = \left(\frac{(1-\tau_k)\gamma}{\psi}\right)^{\frac{\alpha+\gamma}{\alpha}} \left(\frac{1-\alpha-\gamma}{w}\right)^{\frac{1-\alpha-\gamma}{\alpha}} zk; \qquad n = \left(\frac{(1-\tau_k)\gamma}{\psi}\right)^{\frac{\gamma}{\alpha}} \left(\frac{1-\alpha-\gamma}{w}\right)^{\frac{1-\gamma}{\alpha}} zk.$$

Profits are then:

$$\pi(z,k) = (zk)^{\alpha} g(e)^{\gamma} n^{1-\alpha-\gamma} - wn - rk = \left[\underbrace{(\alpha+\gamma)\left(\frac{(1-\tau_k)\gamma}{\psi}\right)^{\frac{\gamma}{\alpha}} \left(\frac{1-\alpha-\gamma}{w}\right)^{\frac{1-\alpha-\gamma}{\alpha}} z - r}_{\pi^{\star}(z)}\right] k.$$

Crucially, profits, labor, and effort are proportional to the level of capital the entrepreneur uses. The entrepreneur will only demand capital and operate their firm if the (after-tax) profits net of the effort cost are positive, that is:

$$k \geq 0 \longleftrightarrow (1-\tau_k)\,\pi^{\star}(z) - \underbrace{\frac{u_e h^{'}(e)}{u_c}}_{\text{Shadow Price}=\psi} \varepsilon(z) \geq 0,$$

where the shadow price of the effort cost is equal to ψ given our assumptions and

$$\varepsilon(z) \equiv \frac{e(z,k)}{k} = \left(\frac{(1-\tau_k)\,\gamma}{\psi}\right)^{\frac{\alpha+\gamma}{\alpha}} \left(\frac{1-\alpha-\gamma}{w}\right)^{\frac{1-\alpha-\gamma}{\alpha}} z.$$

In order to demand capital the entrepreneur must make profits to cover the cost of effort after taxes.

The optimal demand for capital is then:

$$k^{\star}(z,a) = \begin{cases} \lambda a & \text{if } \alpha\left(\frac{(1-\tau_k)\gamma}{\psi}\right)^{\frac{\gamma}{\alpha}}\left(\frac{1-\alpha-\gamma}{w}\right)^{\frac{1-\alpha-\gamma}{\alpha}} z > r \\ [0,\lambda a] & \text{if } \alpha\left(\frac{(1-\tau_k)\gamma}{\psi}\right)^{\frac{\gamma}{\alpha}}\left(\frac{1-\alpha-\gamma}{w}\right)^{\frac{1-\alpha-\gamma}{\alpha}} z = r \\ 0 & \text{if } \alpha\left(\frac{(1-\tau_k)\gamma}{\psi}\right)^{\frac{\gamma}{\alpha}}\left(\frac{1-\alpha-\gamma}{w}\right)^{\frac{1-\alpha-\gamma}{\alpha}} z < r \end{cases} \cdot$$

We then replace $k^{\star}(z,a)$ back and get the optimal profits, effort and labor demand.

Before proceeding to the optimal savings problem, we need to determine the level of the capital demand for entrepreneurs with different productivity levels. The relevant case has high-productivity entrepreneurs demanding $k^{\star}(z_h,a) = \lambda a$ for a total demand of $K_h = \lambda\mu A_h$. The remaining assets are used by the low-productivity entrepreneurs who will be indifferent between any production level. The total demand for capital required to clear the market is $K_L = (1-\mu)\,A_L - (\lambda-1)\,\mu A_h$.

Let $\lambda_{\ell,\iota} \equiv \frac{k_\iota}{a_\iota}$ be the ratio of capital to assets of low-productivity entrepreneur ι, for $\iota \in [\mu,1]$. We show below that the savings choice of the entrepreneur is independent of the value of $\lambda_{\ell,\iota}$.

Entrepreneurial Savings.

$$V_\iota(a,z) = \max_{\left\{c,a^{'}\right\}} \ln\left(c - \psi e_\iota(z,a)\right) + \beta\delta E\left[V_\iota\left(a^{'},z^{'}\right)|z\right] \qquad \text{s.t. } c + a^{'} = R_\iota(z)\,a$$

where $R(z) \equiv (1-\tau_a) + (1-\tau_k)\left(r + \pi^{\star}(z)\,\lambda_\iota(z)\right)$, $e_\iota(z,a) = \varepsilon(z)\,\lambda_\iota(z)\,a$, and

$$\lambda_\iota(z) = \begin{cases} \lambda & \text{if } z = z_h \\ \lambda_{\iota,\ell} & \text{if } z = z_\ell. \end{cases}$$

We solve the dynamic programming problem via guess and verify. To this end, we guess that the value function of an entrepreneur with productivity z_i, $i \in \{\ell,h\}$, has the form

$$V_{i,\iota}(a) = m_{i,\iota} + n\log(a)\,,$$

where $\{m_{\ell,\iota}, m_{h,\iota}\}_{\iota\in\{0,1\}}, n \in \mathbb{R}$ are coefficients. Under this guess, the optimal savings choice of the entrepreneur is characterized by

$$\frac{1}{(R_{i,\iota} - \psi\varepsilon_i\lambda_{i,\iota})\, a - a_i^{'}} = \frac{\beta\delta n}{a_i^{'}} \longrightarrow a_i^{'} = \frac{\beta\delta n}{1+\beta\delta n}(R_{i,\iota} - \psi\varepsilon_i\lambda_{i,\iota})\, a.$$

Replacing the savings rule into the value function gives:

$$m_{i,\iota} + n\log(a) = \log\left((R_{i,\iota} - \psi\varepsilon_i\lambda_{i,\iota})\, a - a_i^{'}\right) + \beta\delta m_{i,\iota} + \beta\delta n \log\left(a_i^{'}\right)$$

Matching coefficients:

$$\begin{aligned} n &= 1 + \beta\delta n \\ m_{i,\iota} &= \beta\delta n \log(\beta\delta n) + (1+\beta\delta n)\log\left(\frac{R_{i,\iota} - \psi\varepsilon_i\lambda_{i,\iota}}{1+\beta\delta n}\right) + \beta\delta m_{i,\iota}. \end{aligned}$$

This delivers the optimal saving decision of the entrepreneur:

$$a' = \beta\delta\left(R_\iota(z) - \psi\varepsilon(z)\lambda_\iota(z)\right) a.$$

Finally, we solve for the remaining coefficients. When high-productivity entrepreneurs are constrained and low-productivity entrepreneurs are indifferent between any level of production it holds that returns are independent of the identity of the entrepreneur, so that

$$R_\iota(z) - \psi\varepsilon(z)\lambda_\iota(z) = R(z) - \psi\varepsilon(z)\lambda \equiv \hat{R}(z).$$

This allows us to solve for m_ℓ and m_h as:

$$m_i = \frac{1}{(1-\beta\delta)^2}\left(\log\left((\beta\delta)^{\beta\delta}(1-\beta\delta)^{1-\beta\delta}\right)\right) + \frac{1}{(1-\beta\delta)^2}\log\hat{R}(z).$$

D.2 Equilibrium and Aggregation

In equilibrium the interest rate is such that the low-productivity entrepreneurs are indifferent between lending their assets or using them in their own firm. Lending the assets gives them a (before-tax) return of r, using them gives them $\pi^\star(z_\ell)$ but it also entails a utility cost because of effort, which we know from the previous results is proportional to assets, same as returns and profits. The agents will be indifferent if the (after-tax) profits net of effort costs are zero:

$$0 = (1-\tau_k)\,\pi^\star(z_\ell) - \underbrace{\frac{u_e h^{'}(e)}{u_c}}_{\text{Shadow Price}=\psi}\varepsilon(z_\ell).$$

Replacing for the optimal solution of the entrepreneur's problem:

$$r = \alpha \left(\frac{(1-\tau_k)\,\gamma}{\psi} \right)^{\frac{\gamma}{\alpha}} \left(\frac{1-\alpha-\gamma}{w} \right)^{\frac{1-\alpha-\gamma}{\alpha}} z_\ell$$

We can then exploit the linearity of the savings function to aggregate.

Lemma 7. *If $s_h < 1/\lambda$, output, wages, interest rate, and gross returns on savings are:*

$$\begin{aligned}
Y &= \left(\frac{(1-\tau_k)\,\gamma}{\psi} \right)^{\frac{\gamma}{1-\gamma}} (ZK)^{\frac{\alpha}{1-\gamma}} L^{\frac{1-\alpha-\gamma}{1-\gamma}} \\
E &= \left(\frac{(1-\tau_k)\,\gamma}{\psi} \right)^{\frac{1}{1-\gamma}} (ZK)^{\frac{\alpha}{1-\gamma}} L^{\frac{1-\alpha-\gamma}{1-\gamma}} \\
w &= (1-\alpha-\gamma) \left(\frac{(1-\tau_k)\,\gamma}{\psi} \right)^{\frac{\gamma}{1-\gamma}} \left(\frac{ZK}{L} \right)^{\frac{\alpha}{1-\gamma}} \\
r &= \alpha \left(\frac{(1-\tau_k)\,\gamma}{\psi} \right)^{\frac{\gamma}{1-\gamma}} \left(\frac{L}{ZK} \right)^{\frac{1-\alpha-\gamma}{1-\gamma}} z_\ell \\
R_\ell &= (1-\tau_a) + (1-\tau_k) \left(\frac{(1-\tau_k)\,\gamma}{\psi} \right)^{\frac{\gamma}{1-\gamma}} \left(\frac{L}{ZK} \right)^{\frac{1-\alpha-\gamma}{1-\gamma}} (\alpha + \gamma\lambda_\iota)\, z_\ell \\
R_h &= (1-\tau_a) + (1-\tau_k) \left(\frac{(1-\tau_k)\,\gamma}{\psi} \right)^{\frac{\gamma}{1-\gamma}} \left(\frac{L}{ZK} \right)^{\frac{1-\alpha-\gamma}{1-\gamma}} (\alpha z_\lambda + \gamma\lambda z_h)
\end{aligned}$$

and

$$\hat{R}(z) \equiv R(z) - \psi\varepsilon(z)\,\lambda = \begin{cases} (1-\tau_a) + (1-\tau_k)\,\alpha \left(\frac{(1-\tau_k)\gamma}{\psi} \right)^{\frac{\gamma}{1-\gamma}} \left(\frac{L}{ZK} \right)^{\frac{1-\alpha-\gamma}{1-\gamma}} z_\ell & \text{if } z = z_\ell \\ (1-\tau_a) + (1-\tau_k)\,\alpha \left(\frac{(1-\tau_k)\gamma}{\psi} \right)^{\frac{\gamma}{1-\gamma}} \left(\frac{L}{ZK} \right)^{\frac{1-\alpha-\gamma}{1-\gamma}} z_\lambda & \text{if } z = z_h \end{cases}$$

Proof. We start by considering the labor market clearing condition, we get

$$\begin{aligned}
n^\star (z_h, K_h) + n^\star (z_\ell, K_\ell) &= L; \\
\left(\frac{(1-\tau_k)\,\gamma}{\psi} \right)^{\frac{\gamma}{\alpha}} \left(\frac{1-\alpha-\gamma}{w} \right)^{\frac{1-\gamma}{\alpha}} (z_h K_h + z_\ell K_\ell) &= L; \\
(1-\alpha-\gamma) \left(\frac{(1-\tau_k)\,\gamma}{\psi} \right)^{\frac{\gamma}{1-\gamma}} \left(\frac{ZK}{L} \right)^{\frac{\alpha}{1-\gamma}} &= w.
\end{aligned}$$

Using this condition for wages we obtain an expression for the interest rate,

$$r = \alpha \left(\frac{(1-\tau_k)\,\gamma}{\psi} \right)^{\frac{\gamma}{1-\gamma}} \left(\frac{L}{ZK} \right)^{\frac{1-\alpha-\gamma}{1-\gamma}} z_\ell,$$

and for total effort,

$$\left(\frac{E}{ZK}\right)^{\alpha} = \left(\frac{(1-\tau_k)\gamma}{\psi}\right)^{\alpha+\gamma}\left(\frac{1-\alpha-\gamma}{w}\right)^{1-\alpha-\gamma};$$
$$E = \left(\frac{(1-\tau_k)\gamma}{\psi}\right)^{\frac{1}{1-\gamma}}(ZK)^{\frac{\alpha}{1-\gamma}}L^{\frac{1-\alpha-\gamma}{1-\gamma}}.$$

We can use this expression to get the usual Cobb-Douglas expressions for w and r:

$$w = (1-\alpha-\gamma)\frac{(ZK)^{\alpha}E^{\gamma}L^{1-\gamma-\alpha}}{L}; \qquad r = \alpha\frac{(ZK)^{\alpha}E^{\gamma}L^{1-\gamma-\alpha}}{ZK}z_{\ell}.$$

These two expressions also let us rewrite the profit rate (of capital) of entrepreneurs:

$$\pi^{\star}(z) = \left(\frac{(1-\tau_k)\gamma}{\psi}\right)^{\frac{\gamma}{1-\gamma}}\left(\frac{L}{ZK}\right)^{\frac{1-\alpha-\gamma}{1-\gamma}}(\alpha(z-z_{\ell})+\gamma z) > 0.$$

Profits are positive for both types of entrepreneurs, reflecting the effort costs.

We use the equilibrium profit rates of entrepreneurs to rewrite the gross returns,

$$\begin{aligned}
R(z) &= (1-\tau_a)+(1-\tau_k)(r+\pi^{\star}(z)\lambda);\\
&= (1-\tau_a)+(1-\tau_k)\left(\frac{(1-\tau_k)\gamma}{\psi}\right)^{\frac{\gamma}{1-\gamma}}\left(\frac{L}{ZK}\right)^{\frac{1-\alpha-\gamma}{1-\gamma}}(\alpha(z_{\ell}+\lambda(z-z_{\ell}))+\gamma\lambda z);\\
&= \begin{cases}(1-\tau_a)+(1-\tau_k)\left(\frac{(1-\tau_k)\gamma}{\psi}\right)^{\frac{\gamma}{1-\gamma}}\left(\frac{L}{ZK}\right)^{\frac{1-\alpha-\gamma}{1-\gamma}}(\alpha+\gamma\lambda)z_{\ell} & \text{if } z=z_{\ell}\\ (1-\tau_a)+(1-\tau_k)\left(\frac{(1-\tau_k)\gamma}{\psi}\right)^{\frac{\gamma}{1-\gamma}}\left(\frac{L}{ZK}\right)^{\frac{1-\alpha-\gamma}{1-\gamma}}(\alpha z_{\lambda}+\gamma\lambda z_h) & \text{if } z=z_h\end{cases}.
\end{aligned}$$

The return net of effort cost is

$$\begin{aligned}
\hat{R}(z) &= R(z)-\psi\varepsilon(z)\lambda;\\
&= \begin{cases}(1-\tau_a)+(1-\tau_k)\alpha\left(\frac{(1-\tau_k)\gamma}{\psi}\right)^{\frac{\gamma}{1-\gamma}}\left(\frac{L}{ZK}\right)^{\frac{1-\alpha-\gamma}{1-\gamma}}z_{\ell} & \text{if } z=z_{\ell}\\ (1-\tau_a)+(1-\tau_k)\alpha\left(\frac{(1-\tau_k)\gamma}{\psi}\right)^{\frac{\gamma}{1-\gamma}}\left(\frac{L}{ZK}\right)^{\frac{1-\alpha-\gamma}{1-\gamma}}z_{\lambda} & \text{if } z=z_h\end{cases}.
\end{aligned}$$

We aggregate output in terms of total capital using the constant ratio of labor to capital across entrepreneurs. The output of an individual entrepreneur with productivity z and capital k is

$$y(z,k) = \left(\frac{(1-\tau_k)\gamma}{\psi}\right)^{\frac{\gamma}{\alpha}}\left(\frac{1-\alpha-\gamma}{w}\right)^{\frac{1-\alpha-\gamma}{\alpha}}zk.$$

Aggregate output is the sum of the total output produced by all entrepreneurs,

$$Y = \left(\frac{(1-\tau_k)\,\gamma}{\psi}\right)^{\frac{\gamma}{\alpha}} \left(\frac{1-\alpha-\gamma}{w}\right)^{\frac{1-\alpha-\gamma}{\alpha}} (z_h K_h + z_\ell K_\ell) = \left(\frac{(1-\tau_k)\,\gamma}{\psi}\right)^{\frac{\gamma}{1-\gamma}} (ZK)^{\frac{\alpha}{1-\gamma}} L^{\frac{1-\alpha-\gamma}{1-\gamma}}$$

For completeness we also consider the aggregate effort of high- and low-productivity entrepreneurs:

$$E_i \equiv \int e\left(z, k_{\iota,i}\right) d\iota = \varepsilon\left(z_i\right) \int k_{\iota,i} d\iota = \left[\frac{(1-\tau_k)\,\gamma}{\psi} Z K^{-(1-\alpha-\gamma)} L^{1-\alpha-\gamma}\right]^{\frac{1}{1-\gamma}} z_i K_i$$

This completes the derivation of the results.

□

Evolution of aggregates: Using the savings decision rules, we obtain the law of motion for the wealth held by low- and high-productivity entrepreneurs:

$$A_i' = \delta^2 \beta \hat{R}_i A_i + (1-\delta)\,\overline{a},$$

where $\overline{a} \equiv K = (1-\mu)\,A_\ell + \mu A_h$ is the endowment of a newborn entrepreneur, equal to the total (average) wealth in the economy. Combining these we obtain the low of motion of capital:

$$\frac{K'}{K} = \delta^2 \beta \left(s_h \hat{R}_h + (1-s_h)\,\hat{R}_\ell\right) + (1-\delta)\,.$$

Stationary competitive equilibrium and efficiency gains from wealth taxes: It must be that the wealth weighted returns net of effort costs are constant,

$$s_h \hat{R}_h + (1-s_h)\,\hat{R}_\ell = \frac{1}{\beta\delta}$$

$$(1-\tau_a) + (1-\tau_k)\,\alpha \left(\frac{(1-\tau_k)\,\gamma}{\psi}\right)^{\frac{\gamma}{1-\gamma}} \left(\frac{L}{ZK}\right)^{\frac{1-\alpha-\gamma}{1-\gamma}} Z = \frac{1}{\beta\delta}$$

This is similar to the result in (26) but it includes the distortionary effect of capital income taxes on effort. As in Proposition 1, this result implies that returns (now net of effort cost) are:

$$\hat{R}(z) = \begin{cases} (1-\tau_a) + \left(\frac{1}{\beta\delta} - (1-\tau_a)\right)\frac{z_\ell}{Z} & \text{if } z = z_\ell \\ (1-\tau_a) + \left(\frac{1}{\beta\delta} - (1-\tau_a)\right)\frac{z_\lambda}{Z} & \text{if } z = z_h \end{cases}.$$

The equations for the evolution of assets $\left(A_i'\right)$ and the steady state of returns (above) imply that equation (31) applies unchanged and determines the stationary level of productivity as in Section 2.3.

Consequently, Propositions 2 and 3 apply to this economy without modifications:

Proposition 11. *A stationary competitive equilibrium exists and is unique if and only if λ satisfies Assumption 1, and an increase in the wealth tax in such an equilibrium increases productivity Z.*

The difference between the benchmark model (Section 2) and the model with effort is in the response of aggregate variables other than Z to changes in taxes. All directions are maintained, but there is now an additional source of changes on aggregates: a direct effect of taxes on the effort of entrepreneurs. When an increase in wealth taxes reduces capital income taxes, also reduces the distortions on the effort choice of entrepreneurs.

Before establishing the effects of a change in taxes on aggregate variables we revisit the role of government spending. The Government's budget is still given by 24 and Assumption 3 still implies the link between capital income and wealth taxes in equation (45).Then, equilibrium capital is

$$K = \left(\alpha\beta\delta\frac{1-\theta}{1-\beta\delta}\right)^{\frac{1-\gamma}{1-\alpha-\gamma}} \left(\frac{(1-\tau_k)\,\gamma}{\psi}\right)^{\frac{\gamma}{1-\alpha-\gamma}} Z^{\frac{\alpha}{1-\alpha-\gamma}} L.$$

Crucially, capital depends directly on capital income taxes through their effect on effort. Alternatively, we can write the value of capital in terms of the level of the wealth tax:

$$K = (\alpha\beta)^{\frac{1-\gamma}{1-\alpha-\gamma}} \left(\frac{1-\theta}{1-\beta\delta}\right)^{\frac{1}{1-\alpha-\gamma}} \left(\frac{(1-\beta\delta\,(1-\tau_a))\,\gamma}{\psi}\right)^{\frac{\gamma}{1-\alpha-\gamma}} Z^{\frac{\alpha}{1-\alpha-\gamma}} L.$$

This makes it clear that aggregate capital increases with the wealth tax both through the efficiency gains (higher Z) and the decrease in distortions, lower τ_k.

Lemma 8. *If $\tau < \overline{\tau}_a$ and under Assumption 3, an increase in the wealth tax (τ_a) increases aggregate entrepreneurial effort, capital, output, and wages, $\frac{dE}{d\tau_a}, \frac{dK}{d\tau_a}, \frac{dY}{d\tau_a}, \frac{dw}{d\tau_a} > 0$. It also increases the wealth share of high-productivity entrepreneurs, $\frac{ds_h}{d\tau_a} > 0$, and the after-tax return net of effort costs of high-productivity entrepreneurs, $\frac{d\hat{R}_h}{d\tau_a} > 0$, while the after-tax returns net of effort costs of low-productivity entrepreneurs decreases, $\frac{d\hat{R}_\ell}{d\tau_a} < 0$.*

Proof. The wealth share of high-productivity entrepreneurs increases with productivity productivity, see 12. The results for after-tax returns net of effort costs follow from a straightforward modification of Proposition 4 which gives:

$$\frac{d\hat{R}_h}{d\tau_a} > 0 \qquad \text{and} \qquad \frac{d\hat{R}_\ell}{d\tau_a} < 0.$$

Total capital increases with the wealth tax:

$$\frac{d\log K}{d\log\tau_a} = \frac{\gamma}{1-\alpha-\gamma}\frac{\beta\delta\tau_a}{1-\beta\delta\,(1-\tau_a)} + \frac{\alpha}{1-\alpha-\gamma}\frac{d\log Z}{d\log\tau_a} > 0.$$

It follows immediately that output, wages, and total effort increase since they depend positively on ZK and negatively on capital income taxes τ_k.

□

D.3 Optimal Taxes

Entrepreneurial effort changes the choice of optimal taxes in two ways. First, equilibrium aggregates now depend on taxes directly through effort (while before they only changed through changes in productivity). Second, entrepreneurial welfare depends now on after-tax returns net of effort cost. However, only aggregates affect optimal taxes. This is because, in equilibrium, the after-tax returns net of effort cost behave exactly like after-tax returns did in Section 2.

Proposition 12. *Under Assumption 3, there exists a unique tax combination* $\left(\tau_{a,e}^{\star}, \tau_{k,e}^{\star}\right)$ *that maximizes the utilitarian newborn welfare. An interior solution* $\tau_{a,e}^{\star} < \overline{\tau}_a$ *satisfies*

$$\frac{\gamma}{1-\alpha-\gamma}\frac{\beta\delta}{\frac{\log Z}{d\tau_a}} + \frac{\alpha}{1-\alpha-\gamma} = -\frac{1-n_w}{1-\beta\delta}\left(\mu\xi_Z^{\hat{R}_h} + (1-\mu)\,\xi_Z^{\hat{R}_\ell}\right),$$

where $\xi_Z^x \equiv \frac{d\log x}{d\log Z}$ *is the elasticity of variable* x *with respect to* Z*. Moreover,* $\tau_{a,e}^{\star} > \tau_a^{\star}$*, with* $\tau_a^{\star}$ *being the optimal tax described in Proposition 6.*

Proof. Newborn welfare is

$$\mathcal{W} = \frac{1}{1-\beta\delta}\left(n_w \log\left(w+T\right) + (1-n_w)\log\overline{a}\right) + \frac{1-n_w}{(1-\beta\delta)^2}\left(\mu\log\hat{R}_h + (1-\mu)\log\hat{R}_\ell\right) + v$$

The first order condition of the government's problem is

$$0 = n_w\frac{d\log\left(w+T\right)}{d\tau_a} + (1-n_w)\frac{d\log\overline{a}}{d\tau_a} + \frac{1-n_w}{1-\beta\delta}\left(\mu\frac{d\log\hat{R}_h}{d\tau_a} + (1-\mu)\frac{d\log\hat{R}_\ell}{d\tau_a}\right),$$

where we have, under Assumption 3,

$$\frac{\log\overline{a}}{d\tau_a} = \frac{\log w+T}{d\tau_a} = \frac{\gamma}{1-\alpha-\gamma}\beta\delta + \frac{\alpha}{1-\alpha-\gamma}\frac{\log Z}{d\tau_a};$$

$$\mu\frac{d\log\hat{R}_h}{d\tau_a} + (1-\mu)\frac{d\log\hat{R}_\ell}{d\tau_a} = \left(\mu\xi_{\hat{R}_h}^{Z} + (1-\mu)\,\xi_{\hat{R}_\ell}^{Z}\right)\frac{\log Z}{d\tau_a}.$$

Joining gives

$$0 = \left[\frac{\gamma}{1-\alpha-\gamma}\frac{\beta\delta}{\frac{\log Z}{d\tau_a}} + \frac{\alpha}{1-\alpha-\gamma} - \frac{\alpha}{1-\alpha}\right] + \left[\frac{\alpha}{1-\alpha} + \frac{1-n_w}{1-\beta\delta}\left(\mu\xi_{\hat{R}_h}^{Z} + (1-\mu)\,\xi_{\hat{R}_\ell}^{Z}\right)\right]$$

The second term is the same as in Proposition 10, were the elasticity of capital and workers' income and capital were equal to $\alpha/1-\alpha$. The average elasticity of returns net of effort cost is equal to the elasticity of returns in Proposition 10. The first term is positive because $\frac{\log Z}{d\tau_a} > 0$ and $\frac{\alpha}{1-\alpha-\gamma} \geq \frac{\alpha}{1-\alpha}$. This implies that the optimal wealth tax is weakly higher than in Proposition 10. and it is equal if and only if $\gamma = 0$.

□

E Persistence of Productivity

E.1 Entrepreneurial Problem

Given constant taxes τ_a and τ_k and prices, an entrepreneur's optimal savings problem is

$$V(a,z) \quad = \quad \max_{a'} \log(c) + \beta \sum_{z'} \mathbb{P}(z' \mid z) V(a', z') \qquad \text{s.t. } c + a' = R(z) a,$$

with $R(z)$ as in (9). We solve this problem via guess and verify. To this end, we guess that the value function of an entrepreneur with productivity z_i, $i \in \{\ell, h\}$, has the form

$$V_i(a) = m_i + n \log(a),$$

where $m_\ell, m_h, n \in \mathbb{R}$ are coefficients. Under this guess, the optimal savings choice is the solution to the following first order condition:

$$\frac{1}{R_i a - a_i'} = \frac{\beta n}{a_i'} \qquad \longrightarrow \qquad a_i' = \frac{\beta n}{1+\beta n} R_i a.$$

Replacing the savings rule into the value function gives:

$$\begin{aligned} V_i(a) &= \log\left(R_i a - a_i'\right) + \beta\left(p V_i\left(a_i'\right) + (1-p) V_j\left(a_i'\right)\right) \\ m_i + n\log(a) &= \log\left(R_i a - a_i'\right) + \beta\left(p m_i + (1-p) m_j\right) + \beta n \log\left(a_i'\right) \\ m_i + n\log(a) &= \beta n \log(\beta n) + (1+\beta n)\log\left(\frac{R_i}{1+\beta n}\right) + \beta\left(p m_i + (1-p) m_j\right) + (1+\beta n)\log(a) \end{aligned}$$

Matching coefficients we obtain

$$n = 1 + \beta n \qquad \text{and} \qquad m_i = \beta n \log(\beta n) + (1+\beta n)\log\left(\frac{R_i}{1+\beta n}\right) + \beta\left(p m_i + (1-p) m_j\right),$$

where $j \neq i$. The solution to the first equation implies $n = \frac{1}{1-\beta}$. This in turn delivers the optimal saving decision of the entrepreneur:

$$a' \quad = \quad \beta R(z) a.$$

Finally, we solve for the remaining coefficients from the system of linear equations:

$$m_i = \frac{\beta}{1-\beta}\log\left(\frac{\beta}{1-\beta}\right) + \frac{1}{1-\beta}\log\left((1-\beta) R_i\right) + \beta\left(p m_i + (1-p) m_j\right),$$

for $i, j \in \{\ell, h\}$ and $i \neq j$. The solution is

$$m_i = \frac{\log(1-\beta)}{1-\beta} + \frac{\beta}{(1-\beta)^2}\log(\beta) + \frac{(1-\beta p)\log R_i + \beta(1-p)\log R_j}{(1-\beta)^2\left(1-\beta(2p-1)\right)}.$$

E.2 Stationary Recursive Competitive Equilibrium

We are interested in the equilibrium where the interest rate is determined by the return of low-productivity entrepreneurs. Recall that the transition matrix for entrepreneurial productivity ensures that $\mu = 1/2$. Using the saving rules in equation (79), we derive the law of motion for the aggregate wealth of each group

$$\mu A_h^{'} = p\beta R_h \mu A_h + (1-p)\,\beta R_\ell\,(1-\mu)\,A_\ell,$$
$$(1-\mu)\,A_\ell^{'} = (1-p)\,\beta R_h \mu A_h + p\beta R_\ell\,(1-\mu)\,A_\ell,$$

and for the aggregate capital ($K \equiv (1-\mu)\,A_\ell + \mu A_h$), where $s_h = \mu A_h / K$

$$\frac{K^{'}}{K} = \beta\left(s_h R_h + (1-s_h)\,R_\ell\right).$$

As in Section 2.3 this condition and Lemma 1 imply that

$$\frac{K^{'}}{K} \;=\; \beta\left((1-\tau_a) + (1-\tau_k)\,\alpha Z^{\alpha} K^{\alpha-1} L^{1-\alpha}\right)$$

In the stationary equilibrium a version of Proposition (1) applies:

$$s_h R_h + (1-s_h)\,R_\ell = (1-\tau_a) + (1-\tau_k)\,\alpha Z^{\alpha} K^{\alpha-1} L^{1-\alpha} = \frac{1}{\beta},$$

$$R_\ell = (1-\tau_a) + \left(\frac{1}{\beta} - (1-\tau_a)\right)\frac{z_\ell}{Z} \quad \text{and} \quad R_h = (1-\tau_a) + \left(\frac{1}{\beta} - (1-\tau_a)\right)\frac{z_\lambda}{Z}. \tag{83}$$

Then, we use the law of motion of assets to obtain

$$\frac{(1-p)\,\beta R_\ell}{1-p\beta R_h} = \frac{s_h}{1-s_h} = \frac{1-p\beta R_\ell}{(1-p)\,\beta R_h}. \tag{84}$$

Replacing R_ℓ and R_h using (83) we get a quadratic equation not unlike that in (31):

$$0 = (1-\rho\beta\,(1-\tau_a)) - (1+\rho\,(1-2\beta\,(1-\tau_a)))\left(\frac{\frac{z_h+z_\ell}{2}}{Z}\right) + \rho\,(1-\beta\,(1-\tau_a))\,\frac{z_h z_\ell}{Z^2} = 0.$$

Studying this quadratic equation, we show that there is a unique stationary equilibrium and obtain necessary and sufficient conditions for it to feature heterogeneous returns. Before providing the formal statement of our result, we discuss the logic behind the proof. For $\rho \leq 0$, there is a unique solution. For $\rho > 0$, there are two positive roots. However, only the larger root satisfies $z_\ell < Z < z_\lambda$. Then, there is always a unique equilibrium. For the equilibrium to feature return heterogeneity with $R_h > R_\ell$ it must be that $Z < z_h$. We obtain an upper bound on the collateral constraint parameter, $\overline{\lambda}_\rho$, that guarantees this.

Proposition 14. *There exists a unique stationary competitive equilibrium that features heterogenous returns ($R_h > R_\ell$), characterized by a productivity level $Z \in (z_\ell, z_h)$, if and only if* *the collateral constraint is not "too loose," that is, λ satisfies*

$$\lambda < \overline{\lambda}_{\rho} \equiv 1 + \frac{1-\rho}{1+\rho\left(1-2\left(\beta\left(1-\tau_{a}\right)+\left(1-\beta\left(1-\tau_{a}\right)\right)\frac{z_{\ell}}{z_{h}}\right)\right)}.$$

Moreover, this condition, stated in terms of an upper bound on λ can be restated as an upper bound on the wealth tax, given λ:

$$\lambda < \overline{\lambda}_{\rho} \longleftrightarrow \tau_{a} \leq \overline{\tau}_{a,\rho} \equiv 1 - \frac{1}{\beta\left(1-\frac{z_{\ell}}{z_{h}}\right)}\left[\frac{\left(\lambda-1\right)\left(\rho+1\right)-\left(1-\rho\right)}{2\left(\lambda-1\right)\rho} - \frac{z_{\ell}}{z_{h}}\right].$$

Proof. For the heterogeneous return equilibrium to arise it must be that $\left(\lambda-1\right)\mu A_{h} < \left(1-\mu\right)A_{\ell}$. First, we show that such an equilibrium is unique and then that it exists under conditions on λ. The equilibrium Z corresponds to the largest root of equation (80). Define the function $h\left(z\right)$ as

$$\begin{aligned} h\left(z\right) = \left(1-\beta\left(1-\tau_{a}\right)\left(2p-1\right)\right)z^{2} - \left(z_{\ell}+z_{\lambda}\right)\left(p-\beta\left(1-\tau_{a}\right)\left(2p-1\right)\right)z & \\ + \left(2p-1\right)z_{\ell}z_{\lambda}\left(1-\beta\left(1-\tau_{a}\right)\right) &= 0. \end{aligned}$$

It is easy to show that $h\left(z_{\ell}\right) = \left(1-p\right)z_{\ell}\left(z_{\ell}-z_{\lambda}\right) < 0$ and $h\left(z_{\lambda}\right) = \left(1-p\right)z_{\lambda}\left(z_{\lambda}-z_{\ell}\right) > 0$. Hence, there is a single root satisfying $z_{\ell} < Z < z_{\lambda}$ because $h\left(z\right)$ is a quadratic function.

Next, we prove that $\left(\lambda-1\right)\mu A_{h} < \left(1-\mu\right)A_{\ell}$ (excess supply of funds) iff $\lambda < \overline{\lambda}_{\rho}$. First, we show that $\left(\lambda-1\right)\mu A_{h} < \left(1-\mu\right)A_{\ell}$ iff $Z < z_{h}$. To see this substitute the definition of $Z = \frac{\left(z_{h}+\left(\lambda-1\right)\left(z_{h}-z_{\ell}\right)\right)\mu A_{h}+z_{\ell}\left(1-\mu\right)A_{\ell}}{\mu A_{h}+\left(1-\mu\right)A_{\ell}}$ into $Z < z_{h}$, some algebra gives $\left(\lambda-1\right)\mu A_{h} < \left(1-\mu\right)A_{\ell}$. Second, we derive the condition on λ so that $h\left(z_{h}\right) > 0$ and thus $Z < z_{h}$:

$$h\left(z_{h}\right)/z_{h}^{2} = 1-\left(2p-1\right)\beta\left(1-\tau_{a}\right)-\frac{\left(z_{\ell}+z_{\lambda}\right)}{z_{h}}\left(p-\left(2p-1\right)\beta\left(1-\tau_{a}\right)\right)+\left(2p-1\right)\frac{z_{\ell}z_{\lambda}}{z_{h}^{2}}\left(1-\beta\left(1-\tau_{a}\right)\right).$$

Inserting $z_{\lambda} = z_{h}+\left(\lambda-1\right)\left(z_{h}-z_{\ell}\right)$ and combining the terms that include $\lambda-1$ gives

$$h\left(z_{h}\right)/z_{h}^{2} = \frac{\left(1-p\right)\left(z_{h}-z_{\ell}\right)}{z_{h}} - \frac{\left(\lambda-1\right)\left(z_{h}-z_{\ell}\right)}{z_{h}}\left(p-\left(2p-1\right)\left(\beta\left(1-\tau_{a}\right)+\left(1-\beta\left(1-\tau_{a}\right)\right)\frac{z_{\ell}}{z_{h}}\right)\right).$$

Since $p-\left(2p-1\right)\left(\beta\left(1-\tau_{a}\right)+\left(1-\beta\left(1-\tau_{a}\right)\right)\frac{z_{\ell}}{z_{h}}\right) > 0$ for all p, then, $h\left(z_{h}\right) > 0$ iff $\lambda-1 < \frac{1-p}{p-\left(2p-1\right)\left(\beta\left(1-\tau_{a}\right)+\left(1-\beta\left(1-\tau_{a}\right)\right)\frac{z_{\ell}}{z_{h}}\right)}$. Finally, recall that this equilibrium can only exist if $\lambda \leq 2$ (this gives $K_{\ell} \geq 0$). Inspecting the previous result it is immediate that $\overline{\lambda} \leq 2$ iff $p \geq 1/2$.

□

Proposition 13. ***(Efficiency Gains from Wealth Taxation)*** *For all $\tau_{a} < \overline{\tau}_{a,\rho}$, an increase in the wealth tax (τ_{a}) increases aggregate productivity, $\frac{dZ}{d\tau_{a}} > 0$, if and only if entrepreneurial productivity is persistent, $\rho > 0$.*

Proof. The equilibrium level of Z is given by the solution of $h(Z) = 0$ where $h(z)$ is defined in equation (80). Differentiating $h(z)$ with respect to τ_a gives

$$\begin{aligned} \frac{d}{d\tau_a} h(z) &= (2p-1)\beta z^2 - (2p-1)\beta(z_\ell + z_\lambda) z + (2p-1)\beta z_\ell z_\lambda \\ &= (2p-1)\beta z_\ell z_\lambda (z - z_\ell)(z - z_\lambda). \end{aligned}$$

The equilibrium Z satisfies $z_\ell < Z < z_\lambda$, so $(z - z_\ell)(z - z_\lambda) < 0$. Thus, $\frac{d}{d\tau_a} h(z) < 0$ if and only if $p > 1/2$. Moreover, $\frac{d}{d\tau_a} h(z) < 0$ for all τ_a if $z_\ell < Z < z_\lambda$. Thus, $\frac{dZ}{d\tau_a} > 0$ as long as $\tau_a \leq \overline{\tau}_{a,\rho}$.

□

We now provide additional results that aid in the explanation of Proposition 13.

Lemma 9. ***(Savings Rates and Wealth Shares)*** *For all $\tau_a < \overline{\tau}_{a,\rho}$, the stationary saving rate of high-productivity entrepreneurs is positive and the saving rate of low-productivity entrepreneurs is negative: $\beta R_h > 1 > \beta R_\ell$. Furthermore, $s_h > 1/2$ if and only if $\rho > 0$.*

Proof. We first show that an entrepreneur's gross saving rate satisfies $\beta R_i > 1$ if and only if $z_i > Z$. This follows immediately by substituting R_i's from equation (83):

$$\beta R_i > 1 \longleftrightarrow \beta(1-\tau_a) + (1 - \beta(1-\tau_a)) z_i / Z > 1 \longleftrightarrow z_i > Z.$$

The result then follows because $z_\ell < Z < z_\lambda$.

Now, consider $s_h \geq 1/2$. We know that $s_h = \frac{Z - z_\ell}{z_\lambda - z_\ell}$, so $s_h > 1/2$ is equivalent to $Z > \frac{z_\lambda + z_\ell}{2}$. We can verify if this is the case by evaluating the residual of (80) at $\frac{z_\lambda + z_\ell}{2}$:

$$\begin{aligned} h\left(\frac{z_\lambda + z_\ell}{2}\right) &= -(2p-1)(1-\beta(1-\tau_a))\left(\frac{z_\lambda + z_\ell}{2}\right)^2 + (2p-1)(1-\beta(1-\tau_a)) z_\ell z_\lambda \\ &= -(2p-1)(1-\beta(1-\tau_a))\left[\left(\frac{z_\lambda + z_\ell}{2}\right)^2 - z_\ell z_\lambda\right] \\ &= -(2p-1)(1-\beta(1-\tau_a))\left(\frac{z_\lambda - z_\ell}{2}\right)^2 < 0 \end{aligned}$$

The residual is negative if and only if $p \geq 1/2$, $\rho > 0$. So, $Z > \frac{z_\lambda + z_\ell}{2}$ and thus $s_h > 1/2$ for $p \geq 1/2$.

□

Lemma 10. ***(Wealth Shares and Returns)*** *For all $\tau_a < \overline{\tau}_{a,\rho}$, the following equations and* *inequalities hold in equilibrium:*

$$s_h = \frac{1-\beta R_\ell}{\beta\left(R_h - R_\ell\right)} = \frac{Z - z_\ell}{z_\lambda - z_\ell} \qquad \frac{ds_h}{dZ} = \frac{1}{z_\lambda - z_\ell} > 0$$
$$R_h = \frac{1}{\beta\left(2p-1\right)}\left(1 - \frac{1-p}{s_h}\right) \qquad \frac{dR_h}{dZ} > 0$$
$$R_\ell = \frac{1}{\beta\left(2p-1\right)}\left(1 - \frac{1-p}{1-s_h}\right) \qquad \frac{dR_h}{dZ} < 0.$$

Moreover, the average returns are always decreasing with productivity, $\frac{d(R_\ell + R_h)}{dZ} < 0$, *and the geometric average of returns decreases,* $\frac{d(R_h R_\ell)}{dZ} < 0$, *if and only if* $\rho > 0$.

Proof. The wealth share is $s_h = \frac{Z - z_\ell}{z_\lambda - z_\ell}$ from the definition of Z in (12), so it is increasing in Z, $\frac{ds_h}{dZ} = \frac{1}{z_\lambda - z_\ell} > 0$. For returns consider the evolution equation for A_h in steady state

$$\left(1 - p\beta R_h\right)\mu A_h \quad = \quad \left(1-p\right)\beta R_\ell \left(1-\mu\right) A_\ell.$$

Manipulating this expression gives

$$R_h = \frac{1}{p\beta} - \left(\frac{1-p}{p}\right)\left(\frac{1-s_h}{s_h}\right) R_\ell.$$

Similarly, from the law of motion for A_ℓ in equilibrium we obtain

$$R_\ell = \frac{1}{p\beta} - \left(\frac{1-p}{p}\right)\left(\frac{s_h}{1-s_h}\right) R_h.$$

Replacing we can solve for R_h and R_ℓ as a function of s_h,

$$R_h = \frac{1}{\beta\left(2p-1\right)}\left(1 - \frac{1-p}{s_h}\right); \qquad R_\ell = \frac{1}{\beta\left(2p-1\right)}\left(1 - \frac{1-p}{1-s_h}\right). \tag{85}$$

Their derivates with respect to Z are,

$$\frac{dR_h}{dZ} = \frac{1-p}{\beta\left(2p-1\right)}\frac{1}{s_h^2}\frac{ds_h}{dZ} > 0; \qquad \frac{dR_\ell}{dZ} = -\frac{\left(1-p\right)}{\beta\left(2p-1\right)}\frac{1}{\left(1-s_h\right)^2}\frac{ds_h}{dZ} < 0. \tag{86}$$

The signs follows from Proposition 13.

Using the results in (85) and (86) we obtain expressions for the derivative of the sum and product of returns with respect to the wealth tax:

$$\frac{d\left(R_h + R_\ell\right)}{dZ} = \frac{-\left(2s_h - 1\right)\left(1-p\right)}{\beta\left(2p-1\right)\left(\left(1-s_h\right)^2 s_h^2\right)}\frac{ds_h}{dZ}; \qquad \frac{d\left(R_h R_\ell\right)}{dZ} = \frac{-\left(2s_h - 1\right)p\left(1-p\right)}{\left[\left(1-s_h\right)s_h\beta\left(2p-1\right)\right]^2}\frac{ds_h}{dZ}$$

$\frac{d(R_h+R_\ell)}{dZ}$ is always negative because $s_h \geq 1/2$ if and only if $p \geq 1/2$, as we proved in the previous Lemma. $\frac{d(R_h R_\ell)}{dZ}$ is negative if and only if $s_h \geq 1/2$, again, this happens if and only if $p \geq 1/2$.

□

E.3 Optimal Taxes

We first discuss the welfare measure we use as the government's objective. Because there is no stationary wealth distribution in the stationary competitive equilibrium of the model, it is not possible to compute aggregate welfare directly. However, it is possible to define policy so as to maximize the welfare change with respect to a benchmark economy. Let B denote the initial benchmark economy with a given level of capital income and wealth taxes and C denote a counterfactual economy with a higher wealth tax and a lower capital income tax, satisfying Assumption 3. Define $\left\{c_t^{\mathrm{j}}(a,i)\right\}$ as the consumption path and $V^{\mathrm{j}}(a,i)$ as the value function of an individual of type $i \in \{w,h,\ell\}$ under economy $\mathrm{j} \in \{\mathrm{B},\mathrm{C}\}$. We ask each individual how much they value being dropped from B to C in terms of a consumption-equivalent welfare measure $\mathrm{CE}_1(a,i)$, which is defined by

$$E\sum_t \beta^{t-1}\log\left(\left(1+\mathrm{CE}_1(a,i)\right)c_t^{\mathrm{B}}(a,i)\right) \quad = \quad E\sum_t \beta^{t-1}\log\left(c_t^{\mathrm{C}}(a,i)\right).$$

Solving for $\mathrm{CE}_1(a,i)$ all terms containing wealth cancel, so we drop wealth from the arguments,

$$\log\left(1+\mathrm{CE}_{1,i}\right) \quad = \quad \begin{cases} \log\left(\frac{w^{\mathrm{C}}+T^{\mathrm{C}}}{w^{\mathrm{B}}+T^{\mathrm{B}}}\right) & \text{if } i=w \\ \frac{(1-\beta)\log\left(\frac{R_i^{\mathrm{C}}}{R_i^{\mathrm{B}}}\right)+\beta(1-p)\left(\log\left(\frac{R_\ell^{\mathrm{C}}}{R_\ell^{\mathrm{B}}}\right)+\log\left(\frac{R_h^{\mathrm{C}}}{R_h^{\mathrm{B}}}\right)\right)}{(1-\beta)(1-\beta(2p-1))} & \text{if } i\in\{\ell,h\}. \end{cases} \tag{87}$$

The aggregate welfare gain is the population-weighted average of welfare gains,

$$\log\left(1+\mathrm{CE}_1\right) = \sum_{i\in\{w,h,\ell\}} n_i \log\left(1+\mathrm{CE}_{1,i}\right), \tag{88}$$

where $n_w \equiv L/(L+1)$ is the population share of workers and $n_h = n_\ell \equiv 1/(L+1)$ the share of entrepreneurs. We also define the average entrepreneurial welfare gain ($\mathrm{CE}_1^{\mathrm{e}}$) as

$$\begin{aligned} \log\left(1+\mathrm{CE}_1^{\mathrm{e}}\right) &= \mu\log\left(1+\mathrm{CE}_{1,h}\right)+(1-\mu)\log\left(1+\mathrm{CE}_{1,\ell}\right) \\ &= \frac{1}{1-\beta}\left(\mu\log\left(\frac{R_h^{\mathrm{C}}}{R_h^{\mathrm{B}}}\right)+(1-\mu)\log\left(\frac{R_\ell^{\mathrm{C}}}{R_\ell^{\mathrm{B}}}\right)\right). \end{aligned} \tag{89}$$

Workers gain from an increase in the wealth tax because their income increase. For entrepreneurs, the welfare effects of the increase in the wealth tax come from changes in after-tax returns. There are two effects. First, a higher wealth tax reduces the current returns of low-productivity entrepreneurs and increase those of high-productivity entrepreneurs. Second, (log-)average of returns decrease with the wealth tax, decreasing entrepreneurs' expectations over future returns and reducing their welfare. The net result of these effects is a lower welfare for the low-productivity entrepreneurs and for entrepreneurs as a group.

The welfare gain for the high-productivity entrepreneurs depends on the magnitude of the decrease in average returns, that in turn depends on the initial return dispersion. There is an upper bound on the dispersion of returns (κ_R) that ensures that the loss from lower expected returns is low relative to the increase in R_h. The upper bound is a function of only β and ρ and does not change with the wealth tax.[36]

Proposition 15. ***(Welfare Gain by Agent Type)*** *For all $\tau_a < \overline{\tau}_{a,\rho}$, if Assumption 3 holds and $\rho > 0$, a marginal increase in the wealth tax (τ_a) increases the welfare of workers ($CE_{1,w} > 0$) and decreases the welfare of low-productivity entrepreneurs ($CE_{1,\ell} < 0$) and the average welfare of entrepreneurs ($CE_1^e < 0$). Furthermore, there exists an upper bound on the dispersion of returns (κ_R) such that an increase in the wealth tax increases the welfare of high-productivity entrepreneurs ($CE_{1,h} > 0$) if and only if $R_h - R_\ell < \kappa_R$.*

Proof. For workers' welfare,

$$\frac{d\log\left(1+\mathrm{CE}_{1,w}\right)}{d\tau_a} = \frac{\alpha}{1-\alpha}\frac{d\log Z}{d\tau_a} > 0 \longleftrightarrow \rho > 0$$

The welfare gain is positive if and only if productivity is persistent because of Proposition 13. The welfare of low-productivity entrepreneurs decreases,

$$\frac{d\log\left(1+\mathrm{CE}_{1,\ell}\right)}{d\tau_a} \propto \left(1-\beta\right)\frac{d\log R_\ell}{d\tau_a} + \beta\left(1-p\right)\frac{d\log R_\ell R_h}{d\tau_a} < 0,$$

following from Lemma (10) $\left(\frac{dR_\ell}{d\tau_a}, \frac{dR_\ell R_h}{d\tau_a} < 0\right)$. The average welfare of entrepreneurs also decreases,

$$\frac{d\log\left(1+\mathrm{CE}_1^{\mathrm{e}}\right)}{d\tau_a} = \frac{\beta\left(1-p\right)}{1-\beta}\frac{1}{R_\ell R_h}\frac{dR_\ell R_h}{d\tau_a} < 0.$$

Finally, for the high-productivity entrepreneurs:

$$\begin{aligned}
\frac{d\log\left(1+\mathrm{CE}_{1,h}\right)}{d\tau_a} &\propto \frac{1-\beta}{R_h}\frac{dR_h}{d\tau_a} + \frac{\beta\left(1-p\right)}{R_\ell R_h}\frac{dR_\ell R_h}{d\tau_a} \\
&= \left[\left(1-\beta\right) - \frac{1}{R_\ell}\frac{\left(2s_h-1\right)p\left(1-p\right)}{\left(1-s_h\right)^2\left(2p-1\right)}\right]\frac{\left(1-p\right)}{\beta\left(2p-1\right)s_h^2 R_h}\frac{ds_h}{d\tau_a} \\
&= \left[\left(1-\beta\right) - \frac{\beta\left(2s_h-1\right)p\left(1-p\right)}{\left(p-s_h\right)\left(1-s_h\right)}\right]\frac{\left(1-p\right)}{\beta\left(2p-1\right)s_h^2 R_h}\frac{ds_h}{d\tau_a}.
\end{aligned}$$

We maintain the assumption that $\rho > 0$, and from Lemma (10) we know that $\frac{ds_h}{d\tau_a} > 0$. So, the

[36]The $\mathrm{CE}_{1,h}$ welfare measure we consider here ignores the effects of the increase in the assets of high-productivity entrepreneurs brought about by the increase in the wealth tax. Taking the change in assets into account makes the welfare change unambiguously positive for them.

sign of derivative of interest depends on the sign of the term in square brackets.

$$\frac{d\log\left(1+\text{CE}_{1,h}\right)}{d\tau_a}\geq 0 \longleftrightarrow 1-\beta\geq\frac{\beta\left(2s_h-1\right)p\left(1-p\right)}{\left(p-s_h\right)\left(1-s_h\right)}.$$

We verify that$s_h < p$ in equilibrium, which together with Lemma (9) implies that the right hand side of the inequality is always positive. To verify that $s_h < p$, note that this condition is equivalent to $Z < pz_\lambda + (1-p)\,z_\ell$, then evaluate function h defined in (80) at $pz_\lambda + (1-p)\,z_\ell$. The value of h is always positive, so it must be that $Z < pz_\lambda + (1-p)\,z_\ell$ and thus $s_h < p$.

Then, the high-type entrepreneurs' welfare gain is positive if and only if

$$g\left(s_h\right)\equiv\left(1-\beta\right)\left(p-s_h\right)\left(1-s_h\right)-\beta\left(2s_h-1\right)p\left(1-p\right)\geq 0. \tag{90}$$

Evaluating at $s_h = 1/2$

$$g\left(s_h\right)=\left(1-\beta\right)\left(p-\frac{1}{2}\right)\frac{1}{2}>0.$$

Evaluating at $s_h = p$

$$g\left(s_h\right)\equiv-\beta\left(2p-1\right)p\left(1-p\right)<0.$$

Moreover, g is continuous for $s_h \in [1/2, p]$ and monotonically decreasing. So, there exists an upper bound $\overline{s}_h$ such that

$$\frac{d\log\left(1+\text{CE}_{1,h}\right)}{d\tau_a}\geq 0 \longleftrightarrow s_h\in\left[\frac{1}{2},\overline{s}_h\right],$$

characterized by the solution to

$$\left(p-\overline{s}_h\right)\left(1-\overline{s}_h\right)-\beta\left(2\overline{s}_h-1\right)p\left(1-p\right)=0.$$

Alternatively, we can make us of the link between s_h and the dispersion of returns:

$$R_h-R_\ell=\frac{\left(1-p\right)\left(2s_h-1\right)}{\beta\left(2p-1\right)\left(1-s_h\right)s_h}.$$

So the high-productivity entrepreneurs benefit from an increase in the wealth tax if and only if the dispersion of returns is low enough:

$$\frac{d\log\left(1+\text{CE}_{1,h}\right)}{d\tau_a}\geq 0 \longleftrightarrow s_h\in\left[\frac{1}{2},\overline{s}_h\right] \longleftrightarrow R_h-R_\ell\in\left[0,\kappa_R\right],$$

where $\kappa_R \equiv \frac{(1-p)(2\overline{s}_h-1)}{\beta(2p-1)(1-\overline{s}_h)\overline{s}_h}$. Note that $\overline{s}_h$, and therefore κ_R, depend only on p and β.

□

We now characterize the optimal tax combination $\left(\tau^\star_{a,\rho}, \tau^\star_{k,\rho}\right)$ that maximizes the utilitarian welfare measure CE_1. Proposition 15 makes clear the key tradeoff when considering the welfare effects of wealth taxation: A higher wealth tax increases the welfare of workers by increasing wages through productivity gains, but they reduce the welfare of entrepreneurs by increasing the dispersion of returns and decreasing their expected value. As we show in Proposition 16 below, the tradeoff is captured by the elasticities of wages and returns to changes in productivity. The welfare gain of workers is proportional to the wage elasticity with respect to productivity, $\xi^{Z}_{w+T} = \frac{\alpha}{1-\alpha}$, while the welfare loss of entrepreneurs is proportional to the average elasticity of returns with respect to productivity, $\mu\xi^{Z}_{R_h} + (1-\mu)\,\xi^{Z}_{R_\ell}$.

Proposition 16. ***(Optimal CE$_1$ Taxes)*** *Under Assumption 3, there exist a unique tax combination $\left(\tau^{\star}_{a,\rho}, \tau^{\star}_{k,\rho}\right)$ that maximizes the utilitarian welfare measure CE$_1$. An interior solution, $\tau^{\star}_{a,\rho} < \overline{\tau}_{a,\rho}$, is the solution to:*

$$0 = n_w \xi^{Z}_{w+T} + \frac{1-n_w}{1-\beta}\left(\mu\xi^{Z}_{R_h} + (1-\mu)\,\xi^{Z}_{R_\ell}\right), \tag{91}$$

where $\xi^{Z}_{x} \equiv \frac{d\log x}{d\log Z}$ is the elasticity of variable x with respect to Z. Furthermore, there exist two cutoff values for α, $\underline{\alpha}_\rho$ and $\overline{\alpha}_\rho$, such that $\left(\tau^{\star}_{a,\rho}, \tau^{\star}_{k,\rho}\right)$ satisfies the following properties:

$$\begin{aligned}
\tau^{\star}_{a,\rho} &\in \left[1-\frac{1}{\beta}, 0\right) \text{ and } \tau^{\star}_{k,\rho} > \theta && \text{if } \alpha < \underline{\alpha}_\rho \\
\tau^{\star}_{a,\rho} &\in \left[0, \frac{\theta(1-\beta)}{\beta(1-\theta)}\right] \text{ and } \tau^{\star}_{k,\rho} \in [0,\theta] && \text{if } \underline{\alpha}_\rho \le \alpha \le \bar{\alpha}_\rho \\
\tau^{\star}_{a,\rho} &\in \left(\frac{\theta(1-\beta)}{\beta(1-\theta)}, \tau^{\max}_{a,\rho}\right) \text{ and } \tau^{\star}_{k,\rho} < 0 && \text{if } \alpha > \bar{\alpha}_\rho,
\end{aligned}$$

where $\underline{\alpha}_\rho$ and $\overline{\alpha}_\rho$ are the solutions to equation (91) with $\tau_a = 0$ and $\tau_a = \tau^{TR} \equiv \frac{\theta(1-\beta)}{\beta(1-\theta)}$, respectively. Recall that $\xi^{Z}_{w+T} = \alpha/1-\alpha$.

As an alternative to CE$_1$, we consider the welfare gain of a stand-in representative low- and high-productivity entrepreneur. We compare the values of the entrepreneurs between being in the Benchmark or Counterfactual economy while holding the average wealth of a low- or high-productivity entrepreneur. We denote this welfare measure as CE$_{2,i}$:

$$\log(1+\mathrm{CE}_{2,i}) = (1-\beta)\left(V^{\mathrm{C}}\left(A^{\mathrm{C}}_i, i\right) - V^{\mathrm{B}}\left(A^{\mathrm{B}}_i, i\right)\right) = \log(1+\mathrm{CE}_{1,i}) + \log\left(A^{\mathrm{C}}_i/A^{\mathrm{B}}_i\right). \tag{92}$$

We can also ask each entrepreneur how much they value being in the counterfactual economy with its average wealth $\left(K^{\mathrm{C}}\right)$ relative to being in the benchmark economy with its average wealth $\left(K^{\mathrm{C}}\right)$. The welfare gain for a type-i entrepreneur is

$$\log\left(1+\widetilde{\mathrm{CE}}_{2,i}\right) = (1-\beta)\left(V^{\mathrm{C}}\left(K^{\mathrm{C}}, i\right) - V^{\mathrm{B}}\left(K^{\mathrm{C}}, i\right)\right) = \log(1+\mathrm{CE}_{1,i}) + \log\left(K^{\mathrm{C}}/K^{\mathrm{B}}\right), \tag{93}$$

and the aggregate (or expected) welfare is

$$\log\left(1+\widetilde{\mathrm{CE}}_{2}\right) = \sum_i n_i \log\left(1+\widetilde{\mathrm{CE}}_{2,i}\right) = \log(1+\mathrm{CE}_{1,i}) + \log\left(K^{\mathrm{C}}/K^{\mathrm{B}}\right). \tag{94}$$

This gives a similar welfare measure to the one used in Section 5. The optimal taxes are similarly given as:

Proposition 17. ***(Optimal $\widetilde{CE}_2$ Taxes)*** *Under Assumption 3, there exist a unique tax combination $\left(\tau^\star_{a,2}, \tau^\star_{k,2}\right)$ that maximizes the utilitarian welfare measure $\widetilde{CE}_2$, an interior solution $\tau^\star_{a,2} < \overline{\tau}_{a,\rho}$ is the solution to:*

$$0 = n_w \xi^Z_{w+T} + (1 - n_w)\,\xi^Z_K + \frac{1 - n_w}{1 - \beta}\left(\mu \xi^Z_{R_h} + (1 - \mu)\,\xi^Z_{R_\ell}\right) \tag{95}$$

where $\xi^Z_x \equiv \frac{d\log x}{d\log Z}$ is the elasticity of variable x with respect to Z. Furthermore, there exist two cutoff values for α, $\underline{\alpha}_2$ and $\overline{\alpha}_2$, such that $\left(\tau^\star_{a,2}, \tau^\star_{k,2}\right)$ satisfies the following properties:

$$\begin{aligned}
\tau^\star_{a,2} &\in \left[1 - \frac{1}{\beta}, 0\right) \text{ and } \tau^\star_{k,2} > \theta && \text{if } \alpha < \underline{\alpha}_2 \\
\tau^\star_{a,2} &\in \left[0, \frac{\theta(1-\beta)}{\beta(1-\theta)}\right] \text{ and } \tau^\star_{k,2} \in [0, \theta] && \text{if } \underline{\alpha}_2 \le \alpha \le \bar{\alpha}_2 \\
\tau^\star_{a,2} &\in \left(\frac{\theta(1-\beta)}{\beta(1-\theta)}, \tau^{\max}_{a,2}\right) \text{ and } \tau^\star_{k,2} < 0 && \text{if } \alpha > \bar{\alpha}_2
\end{aligned}$$

where $\underline{\alpha}_2$ and $\overline{\alpha}_2$ are the solutions to equation (95) with $\tau_a = 0$ and $\tau_a = \frac{\theta(1-\beta)}{\beta(1-\theta)}$, respectively. Recall that $\xi = \alpha/1-\alpha$.

Proof. From (94) we obtain the first order condition to maximize $\widetilde{\mathrm{CE}}_2$:

$$\left[\frac{d\log(1 + \mathrm{CE}_1)}{d\log Z} + (1 - n_w)\frac{d\log K}{d\log Z}\right]\frac{d\log Z}{d\tau_a} = 0$$

$$\left[n_w \xi^Z_{w+T} + \frac{1 - n_w}{1 - \beta}\left(\mu \xi^Z_{R_h} + (1 - \mu)\,\xi^Z_{R_\ell}\right) + (1 - n_w)\,\xi^Z_K\right]\frac{d\log Z}{d\tau_a} = 0$$

As in the proof of Proposition 16 this leads to the optimality condition. Further, we know that $\xi^Z_{w+T} = \xi^Z_K = \alpha/1-\alpha$. The uniqueness of the solution and the definition of the thresholds for α and its implications for the optimal taxes follow from the same arguments as in Proposition 16.

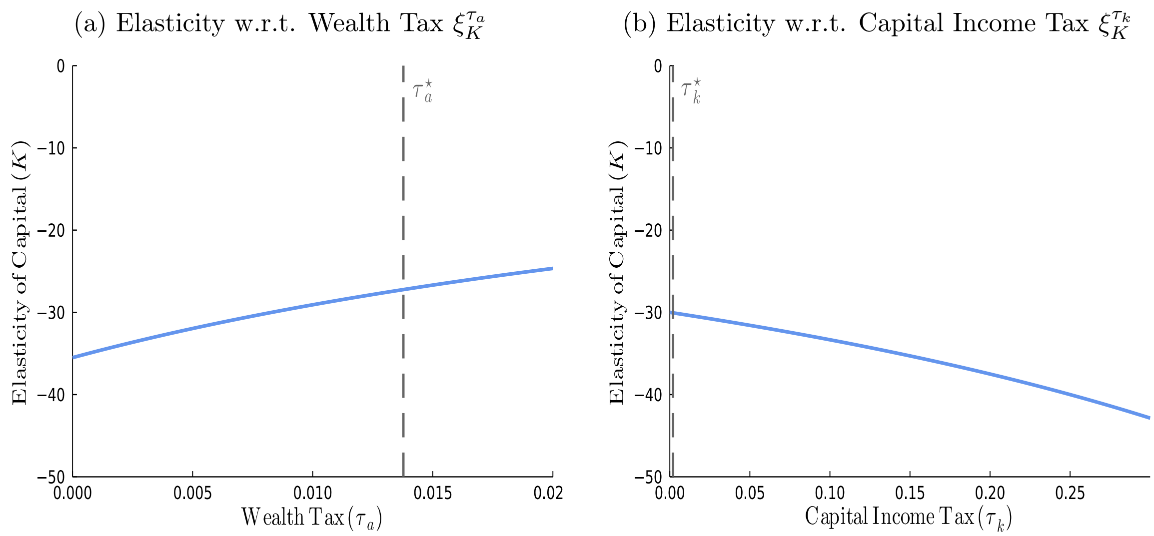
□

Taking into account the role of capital accumulation results in a higher optimal tax level, and lower thresholds $\underline{\alpha}$ and $\overline{\alpha}$:

Corollary 2. ***(Comparison of CE_1 and CE_2 Taxes)*** *The optimal wealth tax is higher when taking the wealth accumulation into account $\left(\tau^\star_{a,2} > \tau^\star_{a,\rho}\right)$. Moreover, the α-thresholds are lower $\underline{\alpha}_2 < \underline{\alpha}_\rho$ and $\overline{\alpha}_2 < \overline{\alpha}_\rho$.*

F Additional Figures

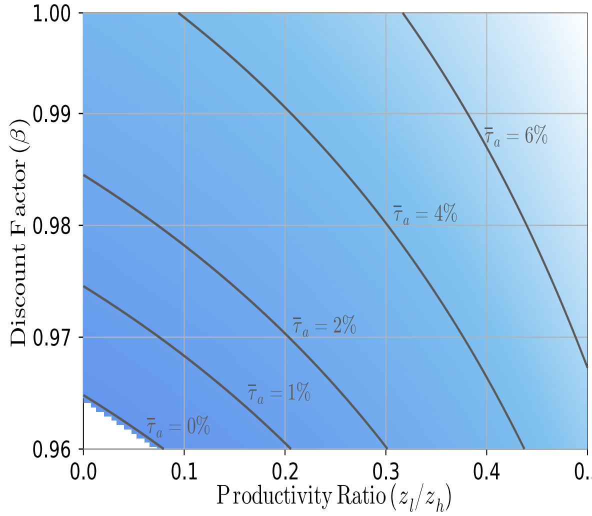
Figure F.2: Capital Elasticity to Taxes



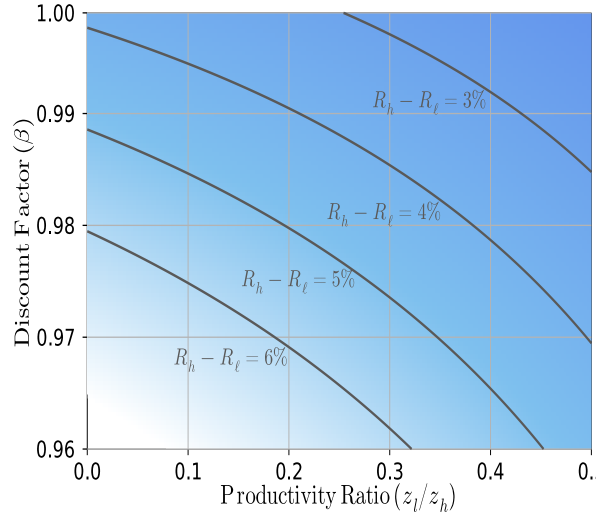
Note: The Figures report the elasticity of aggregate capital, K, to the wealth tax (left) and capital income tax (right). Elasticities are computed as in (32) and (33). To make the magnitudes comparable the elasticity of K with respect to τ_k is re-scaled to reflect the change in capital income taxes that matches a one percentage point change in the wealth tax under Assumption 3. This amount to multiplying (33) by $^{(1-\theta)\beta\delta}/_{(1-\beta\delta)}$. Other parameters are as follows: $\delta = {}^{49}/_{50}$, $\beta\delta = 0.96$, $\mu = 0.10$, $z_h = 1$, $\tau_k = 25\%$, and $\alpha = 0.4$, $\theta = 025$. λ is such that the debt-to-output ratio is 1.5.

Figure F.3: Stationary Equilibrium with Heterogeneous Returns

(a) Upper Bound on Wealth Tax $\overline{\tau}_a$



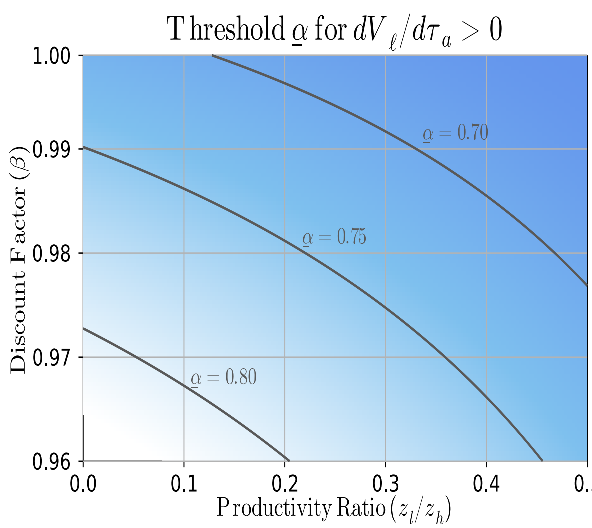
(b) Dispersion of Returns in Equilibrium, $R_h - R_\ell$



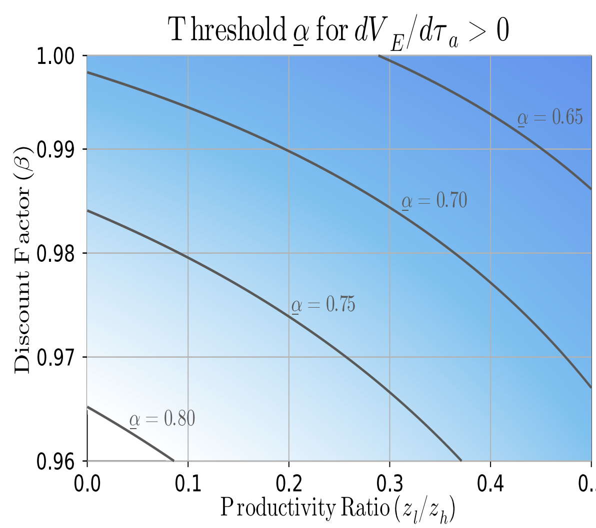
Note: Figure F.3a reports the upper bound on the wealth tax from Corollary 2 for combinations of the discount factor (β) and productivity dispersion (z_ℓ/z_h). Figure F.3b reports the value return dispersion in equilibrium for combinations of the discount factor (β) and productivity dispersion (z_ℓ/z_h). In both figures we set the remaining parameters as follows: $\delta = 49/50$, $\mu = 0.10$, $z_h = 1$, $\tau_k = 25\%$, and $\alpha = 0.4$. λ is such that the debt-to-output ratio in our baseline calibration is 1.5.

Figure F.4: α Thresholds for Entrepreneurial Welfare Gains

(a) Low-Productivity Entrepreneurs: $dV_\ell/d\tau_a > 0$



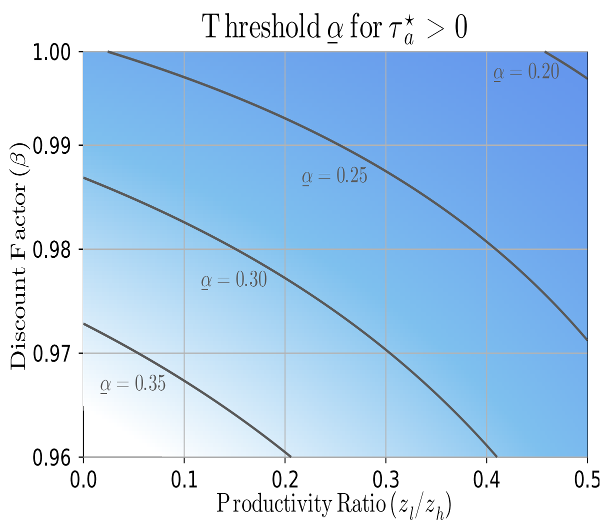
(b) Average Entrepreneur: $dV_E/d\tau_a > 0$



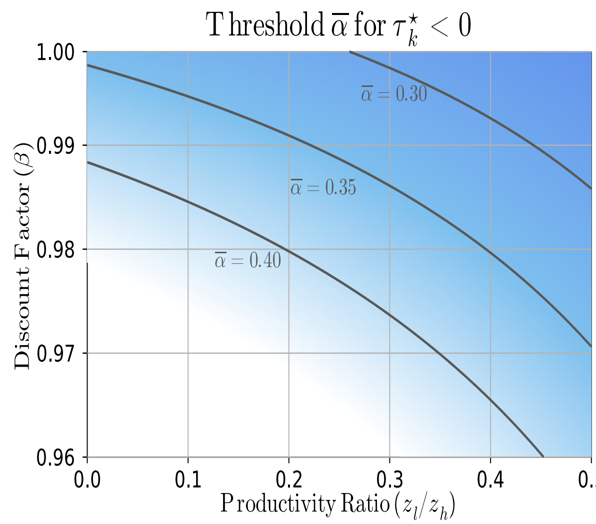
Note: The figures report the threshold value of α above which entrepreneurial welfare increases after an increase in the wealth tax for combinations of the discount factor (β) and productivity dispersion (z_ℓ/z_h). We set the remaining parameters as follows: $\delta = 49/50$, $\mu = 0.10$, $z_h = 1$, $\tau_k = 25\%$, and $\alpha = 0.4$. λ is such that the debt-to-output ratio in our baseline calibration is 1.5.

Figure F.5: α Thresholds for the Optimal Wealth Tax

(a) Lower Threshold $\underline{\alpha}$ for $\tau_a^\star > 0$

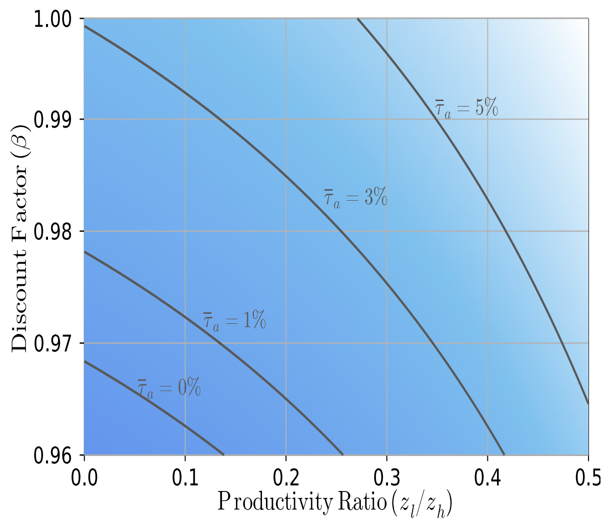


(b) Upper Threshold $\overline{\alpha}$ for $\tau_k^\star < 0$



Note: The figures report the threshold value of α for the optimal wealth tax to be positive (left) and capital income taxes to be positive (right) for combinations of the discount factor (β) and productivity dispersion (z_ℓ/z_h). We set the remaining parameters as follows: $\delta = 49/50$, $\mu = 0.10$, $z_h = 1$, $\tau_k = 25\%$, and $\alpha = 0.4$. λ is such that the debt-to-output ratio in our baseline calibration is 1.5.

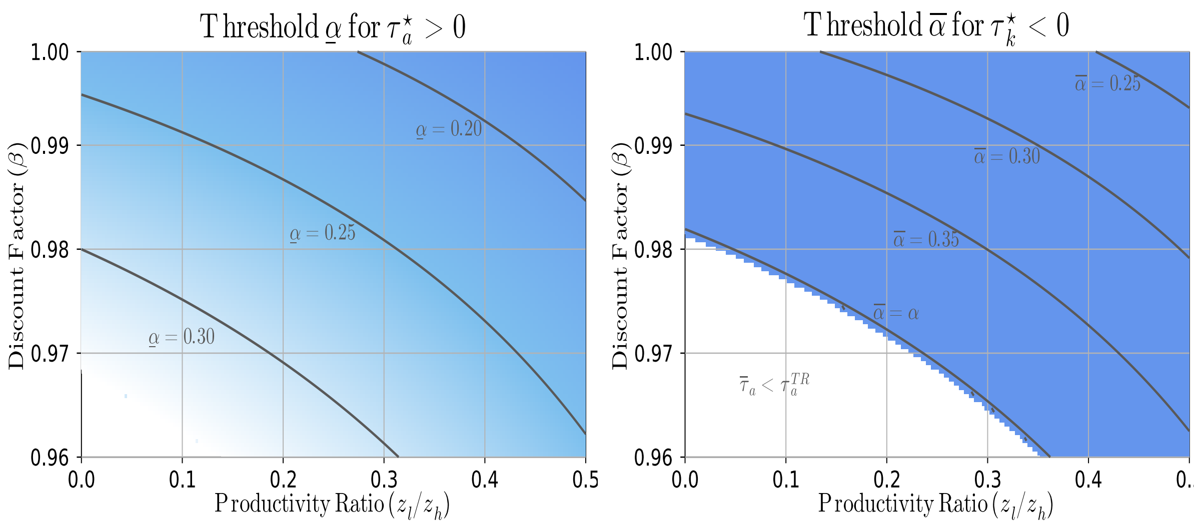
Figure F.6: Upper Bound on the Wealth Tax ($\bar{\tau}_a$) *with Innovation*



Note: The figure reports the upper bound on the wealth tax from Proposition 7 when innovation is endogenous for combinations of the discount factor (β) and productivity dispersion (z_ℓ/z_h). We set the remaining parameters as follows: $\delta = 49/50$, $z_h = 1$, $\tau_k = 25\%$, and $\alpha = 0.4$. λ is such that the debt-to-output ratio in our baseline calibration is 1.5.

Figure F.7: α Thresholds for the Optimal Wealth Tax *with Innovation*

(a) Lower Threshold $\underline{\alpha}$ for $\tau_a^\star > 0$ (b) Upper Threshold $\overline{\alpha}$ for $\tau_k^\star < 0$



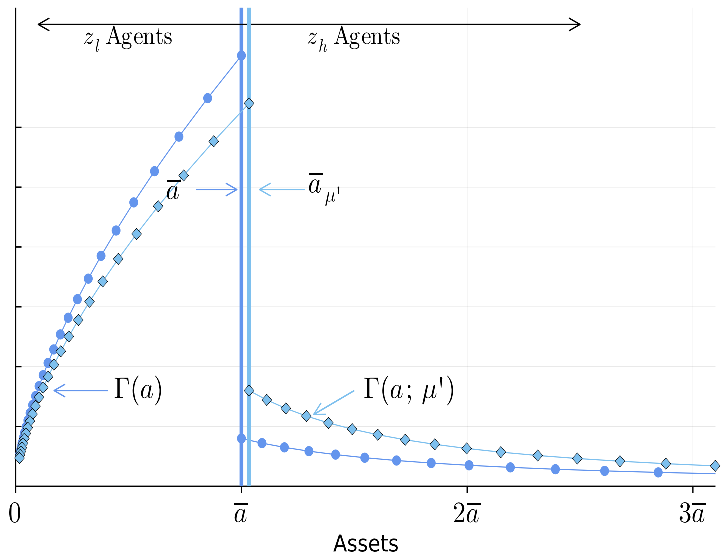
Note: The figures report the threshold value of α for the optimal wealth tax to be positive (left) and capital income tax to be positive (right) when innovation is endogenous for combinations of the discount factor (β) and productivity dispersion (z_ℓ/z_h). We set the remaining parameters as follows: $\delta = 49/50$, $z_h = 1$, $\tau_k = 25\%$, and $\alpha = 0.4$. λ is such that the debt-to-output ratio in our baseline calibration is 1.5.

G Distribution of Wealth

Figure G.8 shows the resulting stationary wealth distribution and how it changes as μ increases. All entrepreneurs start out with wealth $\overline{a}$. The two tails of the distribution depend on the returns of low- and high-productivity entrepreneurs. As μ increases, the average wealth in the economy increases, which raises $\overline{a}$ and shifts the distribution to the right. This shifts all of the mass points and the initial mass of high-productivity entrepreneurs; the mass at all other points is proportional to it. The increase in μ also reduces returns, as shown in the expression for returns in Proposition 4. This means that low-productivity entrepreneurs dissave faster (increasing the distance between mass points) and high-productivity entrepreneurs accumulate assets more slowly (decreasing the distance between mass points) with a higher μ.

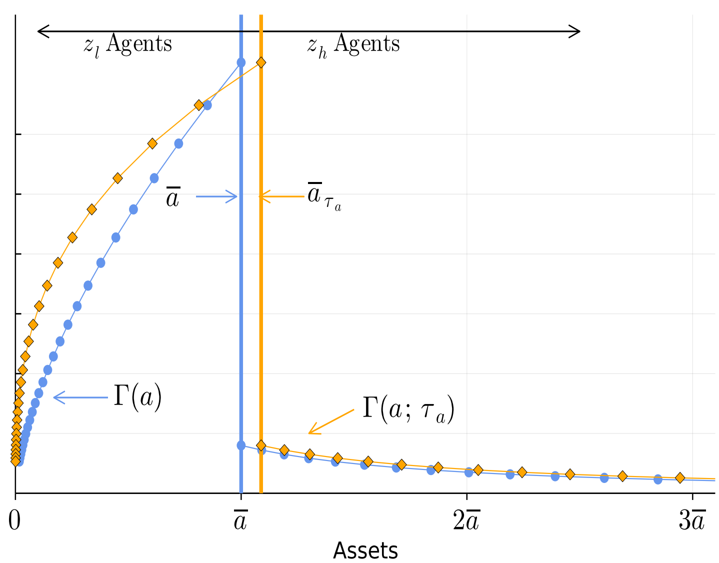
What happens to the wealth distribution when τ_a increases? The whole wealth distribution shifts after an increase in the wealth tax, an outcome that reflects the increase in aggregate wealth and the change in returns. The increase in aggregate capital shifts all mass points to the right, as they are proportional to $\overline{a} = K$. They are further affected by the compounding effect of returns—see Proposition 4. The resulting shift is shown in Figure G.9. When we take into account the changes in innovation effort, there is additional change in the distribution following an increase in the wealth tax. This is because the share of high-productivity entrepreneurs increases, shifting the mass of the distribution towards them, as in Figure G.8.

Figure G.8: Stationary Wealth Distribution



Note: The line marked with circles shows the shape of the stationary wealth distribution for a given μ, with the vertical line indicating the average wealth, $\overline{a}$. The wealth distribution of low-productivity entrepreneurs is to the left of $\overline{a}$ as they dissave and the distribution of high-productivity entrepreneurs is to the right. The line marked with diamonds shows the wealth distribution for a higher level of high-productivity entrepreneurs, $\mu' > \mu$,with a higher average wealth, $\overline{a}_{\mu'}$.

Figure G.9: Stationary Wealth Distribution and Wealth Taxes



Note: The blue line marked with circles is the stationary wealth distribution for an economy with a zero wealth tax, and the orange line with diamonds is the corresponding distribution with a positive wealth tax ($\tau_a > 0$). The vertical lines mark the levels of $\overline{a}$ in the respective economy. The wealth distribution of low-productivity entrepreneurs is to the left of $\overline{a}$ since they dissave and the distribution of high-productivity entrepreneurs is to the right. The wealth tax economy has a higher level of $\overline{a}$ and different mass-points for the distribution as a result. The share of high-productivity entrepreneurs, μ is held constant.