Advance Topics in Macro - 8185 Notes

Sergio Ocampo Díaz

Contents

1 Overview of the course

This course covers solution methods for dynamic general equilibrium models (DSGE). Depending on the characteristics of the model different methods apply to the solution. We understand by the solution of a model a set of policy functions that characterize the optimal response of endogenous variables to changes in exogenous variables and the endogenous states of the model.

The methods presented seek to approximate the solution of the model linearly. These type of methods are much less computationally intensive than global non-linear methods and have high accuracy in most circumstances.

The course starts by presenting solutions to economies without distortions (Part I), the case of the neoclassical growth model is studied along with three different solution methods. Part II deals with economies with distortions in a general way, this is done by studying the prototype "wedge" model of Chari, Kehoe & McGrattan (2008), the authors show that many, more complicated models, can be shown to be equivalent to their prototype. The solution methods presented in Part I are modified to handle the distortions (or wedges).

Then, following Chari, Kehoe & McGrattan (2008), the Business Cycle Accounting procedure is introduced by means of the prototype model of Part II. This procedure allows to determine, given data from national accounts, the relative importance of the different wedges (or distortions), measured by their ability to explain business cycle fluctuations. To better explain the procedure a simple model with news shock in the spirit of Jaimovich & Rebelo (2009) is presented and then its equivalence to the prototype economy is established. The BCA procedure is then explained in two parts, first it is necessary to recover series for all the wedges, this uses the data and the solution to the prototype economy, then the model can be used to decompose the business cycle fluctuations of aggregate variables in the effects of each wedge. Note that in order to conduct the BCA procedure it is first necessary to obtain values for the model's parameters.

Part IV shows how to estimate the model by maximum likelihood using the linear policy functions (of the approximated solution). In order to estimate the model the Kalman filter must be introduced and a way to express the solution of the model in state space established. Once this is done the likelihood of the model is constructed and the model can be estimated.

Finally Part V extends one of the solution methods presented before to deal with economies where agents face signal extraction problems.

Part I
Economies without distortions

The following sections study the stochastic growth model and solve it using four different methods. Section 2 presents the model, its first order conditions and steady state. Section 3 makes a short reference to the dynamic programming problem that solves the model. Section 4 solves for a linear-quadratic approximation of the problem, that is: the objective function is approximated with a quadratic function and the constraints with a linear function, the resulting problem can be solved using dynamic programming (Section 4.1) or by means of Vaughn's method (Section 4.2). The solution in either case consists on policy functions for the endogenous variables of the model. Finally, Section 5 solves for the policy functions directly using the first order conditions of the original model.

2 The stochastic growth model

Consider the following growth model:

$$\begin{aligned} \max_{\{k_{t+1},c_t,x_t,h_t\}} \quad & E\left[\sum_{t=0}^{\infty}\beta^t\left(\log c_t+\psi\log\left(1-h_t\right)\right)N_t\right] \\ \text{s.t.} \quad & 0=k_t^{\alpha}\left(\left(1+\gamma_z\right)^t z_t h_t\right)^{1-\alpha}-c_t-x_t \\ & 0=\left(\left(1-\delta\right)k_t+x_t\right)N_t-N_{t+1}k_{t+1} \end{aligned}$$

Where:

$$\log z_t=\rho\log z_{t-1}+\epsilon_t \qquad \text{and} \qquad \left(1+\gamma_n\right)^t$$

Using the definition of the population level the problem can be stated as:

$$\begin{aligned} \max_{\{\hat{k}_{t+1}\hat{c}_t,\hat{x}_t,h_t\}} \quad & E\left[\sum_{t=0}^{\infty}\left(\beta\left(1+\gamma_n\right)\right)^t\left(\log\hat{c}_t+\psi\log\left(1-h_t\right)+t\log\left(1+\gamma_z\right)\right)\right] \\ \text{s.t.} \quad & 0=\hat{k}_t^{\alpha}\left(z_t h_t\right)^{1-\alpha}-\hat{c}_t-\hat{x}_t \\ & 0=\left(1-\delta\right)\hat{k}_t+\hat{x}_t-\left(1+\gamma_z\right)\left(1+\gamma_n\right)\hat{k}_{t+1} \end{aligned} \tag{2.1}$$

where the hatted variables are defined as:

$$\hat{k}_t=\frac{K_t}{\left(1+\gamma_z\right)^t\left(1+\gamma_n\right)^t}=\frac{k_t}{\left(1+\gamma_z\right)^t}$$

Note also that the consumption and the investment decision can be eliminated using the two constraints in which case the problem is:

$$\max_{\{\hat{k}_{t+1},h_t\}} E\left[\sum_{t=0}^{\infty}\hat{\beta}^t\left(\log\left(\hat{k}_t^{\alpha}\left(z_t h_t\right)^{1-\alpha}+\left(1-\delta\right)\hat{k}_t-\gamma\hat{k}_{t+1}\right)+\psi\log\left(1-h_t\right)+t\log\left(1+\gamma_z\right)\right)\right] \tag{2.2}$$

where $\hat{\beta}=\beta\left(1+\gamma_n\right)$ and $\gamma=\left(1+\gamma_z\right)\left(1+\gamma_n\right)$.

The (non-stochastic) steady state of the model can be obtained from the first order conditions:

$$\begin{aligned} \frac{\left(1-\alpha\right)\hat{k}_t^{\alpha}z_t^{1-\alpha}h_t^{-\alpha}}{\hat{k}_t^{\alpha}\left(z_t h_t\right)^{1-\alpha}+\left(1-\delta\right)\hat{k}_t-\gamma\hat{k}_{t+1}} &= \frac{\psi}{1-h_t} \\ \frac{\gamma}{\hat{k}_t^{\alpha}\left(z_t h_t\right)^{1-\alpha}+\left(1-\delta\right)\hat{k}_t-\gamma\hat{k}_{t+1}} &= \hat{\beta}E\left[\frac{\alpha\hat{k}_{t+1}^{\alpha-1}\left(z_{t+1}h_{t+1}\right)^{1-\alpha}+\left(1-\delta\right)}{\hat{k}_{t+1}^{\alpha}\left(z_{t+1}h_{t+1}\right)^{1-\alpha}+\left(1-\delta\right)\hat{k}_{t+1}-\gamma\hat{k}_{t+2}}\right] \end{aligned}$$

In the non-stochastic steady state $z_t = z_{t+1} = 1$, $\hat{k}_t = \hat{k}_{t+1} = \hat{k}_{t+2} = 0$ and $h_t = h_{t+1} = h$:

$$\begin{aligned}\frac{(1-\alpha)\hat{k}^{\alpha}h^{-\alpha}}{\hat{k}^{\alpha}h^{1-\alpha}+(1-\delta-\gamma)\hat{k}} &= \frac{\psi}{1-h}\\ \gamma &= \hat{\beta}\left(\alpha\hat{k}^{\alpha-1}h^{1-\alpha}+(1-\delta)\right)\end{aligned}$$

From the second equation:

$$\frac{h}{\hat{k}} = \left(\frac{\frac{\gamma}{\hat{\beta}}+\delta-1}{\alpha}\right)^{\frac{1}{1-\alpha}} = \Lambda$$

Replacing in the first equation:

$$\begin{aligned}\frac{(1-\alpha)\Lambda^{-\alpha}}{\left(\Lambda^{1-\alpha}+(1-\delta-\gamma)\right)\hat{k}} &= \frac{\psi}{1-h}\\ (1-h)\frac{\Theta}{\psi} &= \hat{k}\end{aligned}$$

Where $\Theta = \frac{(1-\theta)\Lambda^{-\alpha}}{\left(\Lambda^{1-\alpha}+(1-\delta-\gamma)\right)}$. Then using the definition of Λ:

$$h = \frac{\Lambda\Theta}{\psi}(1-h) \longrightarrow \frac{\psi}{\Lambda\Theta}h = 1-h \longrightarrow h = \left(1+\frac{\psi}{\Lambda\Theta}\right)^{-1}$$

This determines the steady state.

$$(1-h)\frac{\Theta}{\psi}h = \left(1+\frac{\psi}{\Lambda\Theta}\right)^{-1} \qquad \hat{k} = (1-h)\frac{\Theta}{\psi}$$

If $\psi = 0$ then $h = 1$ is the steady state value for labor and

$$\gamma = \hat{\beta}\left(\theta\hat{k}^{\alpha-1}+(1-\delta)\right) \longrightarrow \hat{k} = \left(\frac{\frac{\gamma}{\hat{\beta}}+\delta-1}{\alpha}\right)^{\frac{1}{\alpha-1}}$$

3 Value function iteration

The problem above solves also the functional equation:

$$V_t\left(\hat{k},z\right) = \max_{\left\{\hat{k}_{t+1},h_t\right\}}\left\{\left(\begin{array}{c}\log\left(\hat{k}_t^{\alpha}(z_t h_t)^{1-\alpha}+(1-\delta)\hat{k}_t-\gamma\hat{k}_{t+1}\right)\\ +\psi\log l(1-h_t)+t\log(1+\gamma_z)\end{array}\right)+\hat{\beta}E\left[V_{t+1}\left(\hat{k}_{t+1},z_{t+1}\right)\right]\right\}$$

Note that the time dependence of the value function is deterministic and additively separable from the rest of the function, then one can solve instead for:

$$V\left(\hat{k},z\right) = \max_{\left\{\hat{k}_{t+1},h_t\right\}}\left\{\log\left(\hat{k}_t^{\alpha}(z_t h_t)^{1-\alpha}+(1-\delta)\hat{k}_t-\gamma\hat{k}_{t+1}\right)+\psi\log(1-h_t)+\hat{\beta}E\left[V\left(\hat{k}_{t+1},z_{t+1}\right)\right]\right\}$$

The above problem can be solved by value function iteration.

4 Linear-Quadratic approximation

Let $x_t = \left[z_t, \hat{k}_t\right]'$ be a 2×1 vector of states and $u_t = \left[\hat{k}_{t+1}, h_t\right]$ be a 1×1 vector of control variables.

The one-period return function of the problem in (2.2) is given by:

$$r(x_t, u_t) = r\left(z_t, \hat{k}_t, \hat{k}_{t+1}, h_t\right) = \log\left(\hat{k}_t^{\alpha}(z_t h_t)^{1-\alpha} + (1-\delta)\hat{k}_t - \gamma \hat{k}_{t+1}\right) + \psi \log(1-h_t)$$

A second order approximation around the steady state takes the following form (where $\tilde{x}_{1,t} = x_{i,t} - x_i$ and $\tilde{u}_{i,t} = u_{i,t} - u_i$ denote deviations from steady state):

$$\begin{aligned} r(x_t, u_t) \approx\ & r + r_{x_1}\tilde{x}_{1,t} + r_{x_2}\tilde{x}_{2,t} + r_{u_1}\tilde{u}_{1,t} + r_{u_2}\tilde{u}_{2,t} \\ & + \frac{1}{2} r_{x_1x_1}\tilde{x}_{1,t}^2 + \frac{1}{2} r_{x_2x_2}\tilde{x}_{2,t}^2 + \frac{1}{2} r_{u_1u_1}\tilde{u}_{1,t}^2 + \frac{1}{2} r_{u_2u_2}\tilde{u}_{2,t}^2 \\ & + \frac{1}{2} r_{x_1x_1}\tilde{x}_{1,t}^2 + \frac{1}{2} r_{x_2x_2}\tilde{x}_{2,t}^2 + \frac{1}{2} r_{u_1u_1}\tilde{u}_{1,t}^2 + \frac{1}{2} r_{u_2u_2}\tilde{u}_{2,t}^2 \\ & + r_{x_1x_2}\tilde{x}_{1,t}\tilde{x}_{2,t} + r_{x_1u_1}\tilde{x}_{1,t}\tilde{u}_{1,t} + r_{x_1u_2}\tilde{x}_{1,t}\tilde{u}_{2,t} \\ & + r_{x_2u_1}\tilde{x}_{2,t}\tilde{u}_{1,t} + r_{x_2u_2}\tilde{x}_{2,t}\tilde{u}_{2,t} \\ & + r_{u_1u_2}\tilde{u}_{1,t}\tilde{u}_{2,t} \end{aligned}$$

This can be expressed compactly using matrix notation:

$$\begin{aligned} r(x_t, u_t) \approx\ & r + r_{x_1}\tilde{x}_{1,t} + r_{x_2}\tilde{x}_{2,t} + r_{u_1}\tilde{u}_{1,t} + r_{u_2}\tilde{u}_{2,t} \\ & + \frac{1}{2}\begin{bmatrix} \tilde{x}_{1,t} & \tilde{x}_{2,t} & \tilde{u}_{1,t} & \tilde{u}_{2,t} \end{bmatrix} [H_r] \begin{bmatrix} \tilde{x}_{1,t} \\ \tilde{x}_{2,t} \\ \tilde{u}_{1,t} \\ \tilde{u}_{2,t} \end{bmatrix} \end{aligned}$$

And even more as:

$$\begin{aligned} r(x_t, u_t) & \approx \frac{1}{2}\begin{bmatrix} 1 & \tilde{x}_{1,t} & \tilde{x}_{2,t} & \tilde{u}_{1,t} & \tilde{u}_{2,t} \end{bmatrix} \begin{bmatrix} 2r & \begin{bmatrix} r_{x_1} & r_{x_2} & r_{u_1} & r_{u_2} \end{bmatrix} \\ \begin{bmatrix} r_{x_1} \\ r_{x_2} \\ r_{u_1} \\ r_{u_2} \end{bmatrix} & H_r \end{bmatrix} \begin{bmatrix} 1 \\ \tilde{x}_{1,t} \\ \tilde{x}_{2,t} \\ \tilde{u}_{1,t} \\ \tilde{u}_{2,t} \end{bmatrix} \\ & \approx \frac{1}{2}\begin{bmatrix} 1 & \tilde{x}_{1,t} & \tilde{x}_{2,t} & \tilde{u}_{1,t} & \tilde{u}_{2,t} \end{bmatrix} \begin{bmatrix} 2r & J_r' \\ J_r & H_r \end{bmatrix} \begin{bmatrix} 1 \\ \tilde{x}_{1,t} \\ \tilde{x}_{2,t} \\ \tilde{u}_{1,t} \\ \tilde{u}_{2,t} \end{bmatrix} \\ & \approx \begin{bmatrix} \tilde{x}_t' & \tilde{u}_t' \end{bmatrix} \begin{bmatrix} Q_{3\times 3} & W_{3\times 2} \\ W_{2\times 3}' & R_{2\times 2} \end{bmatrix} \begin{bmatrix} \tilde{x}_t \\ \tilde{u}_t \end{bmatrix} \end{aligned}$$

Where $\tilde{x}_t = [1, \tilde{x}_{1,t}, \tilde{x}_{2,t}]'$ and $\tilde{u}_t = [\tilde{u}_{1,t}, \tilde{u}_{2,t}]'$.

Law of motion for the states is given by:

$$\begin{aligned} \tilde{x}_{t+1} &= A\tilde{x}_t + B\tilde{u}_t + C\epsilon_{t+1} \\ \begin{bmatrix} 1 \\ \tilde{x}_{1,t+1} \\ \tilde{x}_{2,t+1} \end{bmatrix} &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & \rho & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ \tilde{x}_{1,t} \\ \tilde{x}_{2,t} \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \tilde{u}_{1,t} \\ \tilde{u}_{2,t} \end{bmatrix} + \begin{bmatrix} 0 \\ \sigma \\ 0 \end{bmatrix} \epsilon_{t+1} \end{aligned}$$

The problem is then:

$$\max_{\{\tilde{u}_t, \tilde{x}_{t+1}\}} E \sum \beta^t \left(\tilde{x}_t' Q \tilde{x}_t + \tilde{u}_t' R \tilde{u}_t + 2\tilde{x}_t' W \tilde{u}_t\right) \qquad \text{s.t. } \tilde{x}_{t+1} = A\tilde{x}_t + B\tilde{u}_t + C\epsilon_{t+1} \tag{4.1}$$

Next one can re-define variables and matrices to get rid of $\hat{\beta}^t$ and the cross product:

$$\begin{aligned}
\bar{x}_t &= \hat{\beta}^{t/2}\tilde{x}_t \\
\overline{u}_t &= \hat{\beta}^{t/2}\left(\tilde{u}_t + R^{-1}W^{'}\tilde{x}_t\right) \\
\bar{\epsilon}_{t+1} &= \hat{\beta}^{t/2}\epsilon_{t+1} \\
\overline{A} &= \sqrt{\hat{\beta}}\left(A - BR^{-1}W^{'}\right) \\
\overline{B} &= \sqrt{\beta}B \\
\overline{Q} &= Q - WR^{-1}W^{'} \\
\overline{C} &= \sqrt{\beta}C
\end{aligned}$$

The problem is now:

$$\max_{\{\overline{u}_t,\overline{x}_{t+1}\}} E\sum\left(\overline{x}_t^{'}\overline{Q}\overline{x}_t + \overline{u}_t^{'}R\overline{u}_t\right) \qquad \text{s.t. } \overline{x}_{t+1} = \overline{A}\overline{x}_t + \overline{B}\overline{u}_t + \overline{C}\overline{\epsilon}_{t+1} \tag{4.2}$$

4.1 Solution using Ricatti equation

The problem described in equation (4.2) solves as well for the functional equation:

$$V\left(\overline{x}\right) = \max_{\overline{u},\overline{x}_{+1}} \overline{x}^{'}\overline{Q}\overline{x} + \overline{u}^{'}R\overline{u} + EV\left(\overline{x}_{+1}\right) \qquad \text{s.t. } \overline{x}_{+1} = \overline{A}\overline{x} + \overline{B}\overline{u} + \overline{C}\epsilon_{+1} \tag{4.3}$$

This dynamic programming problem can be solved by guess and verify. Guess that the solution has the form:

$$V_t\left(\overline{x}\right) = \overline{x}^{'}P\overline{x} + \hat{\beta}^t c$$

Then the problem is:

$$V_t\left(\overline{x}\right) = \max_{\overline{u},\overline{x}_{+1}} \overline{x}^{'}\overline{Q}\overline{x} + \overline{u}^{'}R\overline{u} + E\left[\overline{x}_{+1}^{'}P\overline{x}_{+1}\right] + \hat{\beta}^{t+1}c \qquad \text{s.t. } \overline{x}_{+1} = \overline{A}\overline{x} + \overline{B}\overline{u} + \overline{C}\overline{\epsilon}_{+1}$$

And replacing the restriction is:

$$\begin{aligned}
V\left(\overline{x}\right) &= \max_{\overline{u},\overline{x}_{+1}} \overline{x}^{'}\overline{Q}\overline{x} + \overline{u}^{'}R\overline{u} + E\left[\left(\overline{A}\overline{x} + \overline{B}\overline{u} + \overline{C}\epsilon_{+1}\right)^{'} P\left(\overline{A}\overline{x} + \overline{B}\overline{u} + \overline{C}\epsilon_{+1}\right)\right] + \hat{\beta}^{t+1}c \\
&= \max_{\overline{u},\overline{x}_{+1}} \overline{x}^{'}\overline{Q}\overline{x} + \overline{u}^{'}R\overline{u} + \overline{x}\overline{A}^{'}P\overline{A}\overline{x} + 2\overline{x}\overline{A}^{'}P\overline{B}\overline{u} + \overline{u}^{'}\overline{B}^{'}P\overline{B}\overline{u} + E\left[\overline{\epsilon}_{+1}^2\right]\overline{C}^{'}P\overline{C} + \hat{\beta}^{t+1}c \\
&= \max_{\overline{u},\overline{x}_{+1}} \overline{x}^{'}\left(\overline{Q} + \overline{A}^{'}P\overline{A}\right)\overline{x} + \overline{u}^{'}\left(R + \overline{B}^{'}P\overline{B}\right)\overline{u} + 2\overline{x}\overline{A}^{'}P\overline{B}\overline{u} + E\left[\overline{\epsilon}_{+1}^2\right]\overline{C}^{'}P\overline{C} + \hat{\beta}^{t+1}c
\end{aligned}$$

The FOC is:

$$\begin{aligned}
2\left(R + \overline{B}^{'}P\overline{B}\right)\overline{u} + 2\overline{B}^{'}P\overline{A}\overline{x} &= 0 \\
\overline{u} &= -\left(R + \overline{B}^{'}P\overline{B}\right)^{-1}\left(\overline{B}^{'}P\overline{A}\right)\overline{x} \\
\overline{u} &= -F\overline{x}
\end{aligned}$$

Note that F is a function of P.

The objective is to find P to obtain F which determines the policy function, relating current states $\bar{x}$ to the decisions taken by the agent $\bar{u}$. Once P and F are known the change of variable is reversed to express the solution in terms of $\tilde{x}$ and $\tilde{u}$.

Replacing the policy function on V one gets:

$$\overline{x}^{'}P\overline{x}+\hat{\beta}^{t}c=\overline{x}^{'}\left(\overline{Q}+\overline{A}^{'}P\overline{A}\right)\overline{x}+\overline{x}^{'}F^{'}\left(R+\overline{B}^{'}P\overline{B}\right)F\overline{x}-2\overline{x}\overline{A}^{'}P\overline{B}F\overline{x}+E\left[\overline{\epsilon}_{+1}^{2}\right]\overline{C}^{'}P\overline{C}+\hat{\beta}^{t+1}c$$

Note that this equation can be further expressed in terms of $\tilde{x}$ and ϵ:

$$\tilde{x}^{'}P\tilde{x}+c=\tilde{x}^{'}\left[\left(\overline{Q}+\overline{A}^{'}P\overline{A}\right)+F^{'}\left(R+\overline{B}^{'}P\overline{B}\right)F-2\overline{A}^{'}P\overline{B}F\right]\tilde{x}+E\left[\epsilon_{+1}^{2}\right]\overline{C}^{'}P\overline{C}+\hat{\beta}c \tag{4.4}$$

Equating coefficients one gets:

$$\begin{aligned} P &= \left(\overline{Q}+\overline{A}^{'}P\overline{A}\right)+F^{'}\left(R+\overline{B}^{'}P\overline{B}\right)F-2\overline{A}^{'}P\overline{B}F \\ P &= \left(\overline{Q}+\overline{A}^{'}P\overline{A}\right)-\overline{A}^{'}P\overline{B}F \end{aligned} \tag{4.5}$$

Where the second step follows from replacing $F^{'}$ in the middle term. This equation is called the Ricatti equation and can be solved by iterating over P, a fixed point gives the value function. Once a value for P has been found the policy function is obtained. The law of motion for the variables is then:

$$\overline{u}_{t} = -F\overline{x}_{t} \tag{4.6}$$

$$\overline{x}_{t+1} = \left(\overline{A}-\overline{B}F\right)\overline{x}_{t}+\overline{C}\epsilon_{t+1} \tag{4.7}$$

Note that this is not the final objective of the method since what is needed is a solution for the dynamics of $\tilde{u}_t$ and $\tilde{x}_t$. Recall that $\overline{x}_t=\hat{\beta}^{t/2}\tilde{x}_t$ and $\overline{u}_t=\hat{\beta}^{t/2}\left(\tilde{u}_t+R^{-1}W^{'}\tilde{x}_t\right)$, replacing on (4.6):

$$\tilde{u}_t=-\left[F+R^{-1}W^{'}\right]\tilde{x}_t \tag{4.8}$$

Equation (4.8) gives the solution for the policy function.

For completeness one can be also interested in the constant c. Equating coefficients from equation (4.4):

$$c=E\left[\epsilon_{+1}^{2}\right]\overline{C}^{'}P\overline{C}+\hat{\beta}c \qquad \longrightarrow \qquad c=\frac{\hat{\beta}}{1-\hat{\beta}}\sigma^{2}C^{'}PC$$

In general if there is more than one shock the system is:

$$\begin{aligned} c &= \hat{\beta}E\left[\epsilon_{+1}^{'}C^{'}PC\epsilon_{+1}\right]+\hat{\beta}c \\ c &= \hat{\beta}E\left[\text{tr}\left(\epsilon_{+1}^{'}C^{'}PC\epsilon_{+1}\right)\right]+\hat{\beta}c \\ c &= \hat{\beta}E\left[\text{tr}\left(\epsilon_{+1}^{'}\epsilon_{+1}C^{'}PC\right)\right]+\hat{\beta}c \\ c &= \hat{\beta}\text{tr}\left(\Sigma C^{'}PC\right)+\hat{\beta}c \\ c &= \frac{\hat{\beta}}{1-\hat{\beta}}\text{tr}\left(\Sigma C^{'}PC\right) \end{aligned}$$

This completely characterizes the problem. The solution to the original problem (equation (4.1)) is given by:

$$V\left(\tilde{x}\right)=\tilde{x}^{'}P\tilde{x}+c$$

4.2 Solution using Vaughan's method

An alternative to the problem presented in section 4.1 is to solve for P using the set of first order conditions of the sequential problem (4.2). This method offers an alternative to solving the Ricatti equation (Eq. 4.5) which requires iteration. Instead P is obtain from the eigen-decomposition of a matrix as shown below.

Recall the problem (4.2) and disregard the stochastic part:

$$\max_{\{\overline{u}_t,\overline{x}_{t+1}\}} \sum \left(\overline{x}_t^{'} \overline{Q} \overline{x}_t + \overline{u}_t^{'} R \overline{u}_t \right) \qquad \text{s.t. } \overline{x}_{t+1} = \overline{A}\overline{x}_t + \overline{B}\overline{u}_t$$

Letting $2\lambda_{t+1}$ be the multiplier on the restriction for $\overline{x}_{t+1}$ one gets:

$$\begin{aligned} 2R\overline{u}_t &= -2\overline{B}^{'}\lambda_{t+1} \\ 2\overline{Q}\overline{x}_{t+1} &= 2\lambda_{t+1} - 2\overline{A}^{'}\lambda_{t+2} \\ \overline{x}_{t+1} &= \overline{A}\overline{x}_t + \overline{B}\overline{u}_t \end{aligned}$$

The system can be reduced by eliminating $\overline{u}$ using the first equation and lagging one period the second. The result is:

$$\overline{Q}\overline{x}_t + \overline{A}^{'}\lambda_{t+1} = \lambda_t \tag{4.9}$$

$$\overline{x}_{t+1} + \overline{B}R^{-1}\overline{B}^{'}\lambda_{t+1} = \overline{A}\overline{x}_t \tag{4.10}$$

Where $\overline{u}_t = -R^{-1}\overline{B}^{'}\lambda_{t+1}$. The objective is to relate $\overline{\lambda}_{t+1}$ to $\bar{x}_t$ in order to obtain the policy function for $\bar{u}$, then the change of variable is reversed, as before, to obtain a relation between decisions $\tilde{u}$ and states $\tilde{x}$.

If $\overline{A}$ is invertible one can express the system (4.9)-(4.10) as:

$$\begin{aligned} \overline{Q}\overline{x}_t + \overline{A}^{'}\lambda_{t+1} &= \lambda_t \\ \overline{A}^{-1}\overline{x}_{t+1} + \overline{A}^{-1}\overline{B}R^{-1}\overline{B}^{'}\lambda_{t+1} &= \overline{x}_t \end{aligned}$$

And then:

$$\begin{aligned} \overline{QA}^{-1}\overline{x}_{t+1} + \left(\overline{QA}^{-1}\overline{B}R^{-1}\overline{B}^{'} + \overline{A}^{'}\right)\lambda_{t+1} &= \lambda_t \\ \overline{A}^{-1}\overline{x}_{t+1} + \overline{A}^{-1}\overline{B}R^{-1}\overline{B}^{'}\lambda_{t+1} &= \overline{x}_t \end{aligned}$$

This forms Vaughan's hamiltonian:

$$\begin{aligned} \begin{bmatrix} \bar{x}_t \\ \lambda_t \end{bmatrix} &= \begin{bmatrix} \overline{A}^{-1} & \overline{A}^{-1}\overline{B}R^{-1}\overline{B}^{'} \\ \overline{QA}^{-1} & \overline{QA}^{-1}\overline{B}R^{-1}\overline{B}^{'} + \overline{A}^{'} \end{bmatrix} \begin{bmatrix} \bar{x}_{t+1} \\ \lambda_{t+1} \end{bmatrix} \\ \begin{bmatrix} \bar{x}_t \\ \lambda_t \end{bmatrix} &= H \begin{bmatrix} \bar{x}_{t+1} \\ \lambda_{t+1} \end{bmatrix} \end{aligned}$$

One can then use the eigen-decomposition of matrix H to get:

$$H = V\Lambda V^{-1}$$

where the columns of V contain the eigenvectors of H and Λ is a diagonal matrix with the corresponding eigenvalues. V can be used to get P.

It is proven that the eigenvalues of H come in reciprocal pairs. Without loss of generality one can write the following:

$$H = \begin{bmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{bmatrix} \begin{bmatrix} \Lambda & 0 \\ 0 & \Lambda^{-1} \end{bmatrix} \begin{bmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{bmatrix}^{-1}$$

where all eigenvalues in Λ are outside of the unit circle. Then:

$$P = V_{21} V_{11}^{-1} \tag{4.11}$$

Once P is obtained one can use equation 4.8 above to characterize the policy function.

Above it was assumed that $\overline{A}$ was invertible, if this doesn't hold the system can be written (directly) as:

$$\begin{aligned} \overline{x}_{t+1} + \overline{B} R^{-1} \overline{B}^{'} \lambda_{t+1} &= \overline{A}\overline{x}_t \\ \overline{A}^{'} \lambda_{t+1} &= \lambda_t - \overline{Q}\overline{x}_t \end{aligned}$$

And in matrix form:

$$\begin{aligned} \begin{bmatrix} \overline{A} & 0 \\ -\overline{Q} & I \end{bmatrix} \begin{bmatrix} \overline{x}_t \\ \lambda_t \end{bmatrix} &= \begin{bmatrix} I & \overline{B} R^{-1} \overline{B}^{'} \\ 0 & \overline{A}^{'} \end{bmatrix} \begin{bmatrix} \overline{x}_{t+1} \\ \lambda_{t+1} \end{bmatrix} \\ H_1 \begin{bmatrix} \overline{x}_t \\ \lambda_t \end{bmatrix} &= H_2 \begin{bmatrix} \overline{x}_{t+1} \\ \lambda_{t+1} \end{bmatrix} \end{aligned}$$

The generalized eigenvalues of the system give the matrix:

$$H = V \Lambda V^{-1}$$

One can then use V as before to obtain P and the policy function.

5 First order conditions method

The previous two methods solve for the policy function as a byproduct of the (more general) dynamic programming problem. An alternative is to solve directly for the policy function using the first order conditions of the sequential problem. This is possible since, provided standard convexity conditions, the solution of the problem is characterized by sequences that satisfy the FOC at all time.

This method requires to obtain the system of first order conditions, linearize it around the steady state and then solve for a linear policy function. It also allows for more flexibility in terms of the variables that can be included. Implicit in the previous methods was the need to eliminate for 'irrelevant' variables, that is variables for which restrictions can be used to replace them out of the problem, like consumption or investment.

5.1 First order conditions

Recall problem (2.1):

$$\begin{aligned} \max_{\left\{\hat{k}_{t+1}\hat{c}_t,\hat{x}_t,h_t\right\}} \quad & E\left[\sum_{t=0}^{\infty}\hat{\beta}^t\left(\log\hat{c}_t+\psi\log\left(1-h_t\right)+t\log\left(1+\gamma_z\right)\right)\right] \\ \text{s.t.} \quad & 0=\hat{k}_t^{\alpha}\left(z_th_t\right)^{1-\alpha}-\hat{c}_t-\hat{x}_t \\ & 0=\left(1-\delta\right)\hat{k}_t+\hat{x}_t-\gamma\hat{k}_{t+1} \end{aligned}$$

Letting $\hat{\beta}^t\lambda_t$ be the multiplier in the first restriction and $\hat{\beta}^t\mu_t$ the multiplier in the second one the set of first order conditions is:

$$\begin{aligned} 0 &= \frac{1}{\hat{c}_t}-\lambda_t \\ 0 &= -\frac{\psi}{1-h_t}+\lambda_t\left(1-\alpha\right)\hat{k}_t^{\alpha}z_t^{1-\alpha}h_t^{-\alpha} \\ 0 &= -\gamma\mu_t+\beta E\left[\lambda_{t+1}\alpha\hat{k}_{t+1}^{\alpha-1}\left(z_{t+1}h_{t+1}\right)^{1-\alpha}+\mu_{t+1}\left(1-\delta\right)\right] \\ 0 &= -\lambda_t+\mu_t \\ 0 &= \hat{k}_t^{\theta}\left(z_th_t\right)^{1-\theta}-\hat{c}_t-\hat{x}_t \\ 0 &= \left(1-\delta\right)\hat{k}_t+\hat{x}_t-\gamma\hat{k}_{t+1} \end{aligned}$$

Although not necessary one can eliminate the multipliers from the system. This leaves:

$$\begin{aligned} 0 &= -\frac{\psi}{1-h_t}+\frac{1}{\hat{c}_t}\left(1-\alpha\right)\hat{k}_t^{\alpha}z_t^{1-\alpha}h_t^{-\alpha} \\ 0 &= -\gamma\frac{1}{\hat{c}_t}+\beta E\left[\frac{1}{\hat{c}_{t+1}}\left(\alpha\hat{k}_{t+1}^{\alpha-1}\left(z_{t+1}h_{t+1}\right)^{1-\alpha}+1-\delta\right)\right] \\ 0 &= \hat{k}_t^{\theta}\left(z_th_t\right)^{1-\theta}-\hat{c}_t-\hat{x}_t \\ 0 &= \left(1-\delta\right)\hat{k}_t+\hat{x}_t-\gamma\hat{k}_{t+1} \end{aligned}$$

Without counting the exogenous variable z_t this is a system of four equations in four variables: $\left[\hat{k},h,\hat{c},\hat{x}\right]$. One can group these variables into endogenous states and controls (o decisions). The endogenous state of the problem is $x=\left[\hat{k}\right]$ and the decisions are $d=\left[h,\hat{c},\hat{x}\right]'$. All exogenous variables are state as well and are group in $S=\left[z_t\right]$. This notation will be used later.

In general one can express the system of first order conditions as a function that maps realizations of the variables in t and $t+1$ to $\mathbb{R}^x \times \mathbb{R}^d$. Call the FOC function f, then we in this model:

$$f\left(x_t, d_t, s_t, x_{t+1}, d_{t+1}, s_{t+1}\right) = \begin{bmatrix} -\frac{\psi}{1-h_t} + \frac{1}{\hat{c}_t}\left(1-\alpha\right)\hat{k}_t^{\alpha} z_t^{1-\alpha} h_t^{-\alpha} \\ -\gamma\frac{1}{\hat{c}_t} + \beta E\left[\frac{1}{\hat{c}_{t+1}}\left(\alpha\hat{k}_{t+1}^{\alpha-1}\left(z_{t+1}h_{t+1}\right)^{1-\alpha} + 1 - \delta\right)\right] \\ \hat{k}_t^{\theta}\left(z_t h_t\right)^{1-\theta} - \hat{c}_t - \hat{x}_t \\ \left(1-\delta\right)\hat{k}_t + \hat{x}_t - \gamma\hat{k}_{t+1} \end{bmatrix}$$

A solution for this problem is a pair of functions $h_x(x, S)$ and $h_d(x, S)$ so that if $x_{t+1} = h_x(x_t, S_t)$ and $d_t = h_d(x_t, S_t)$ then $f(\cdot) = 0$ at all times for any pair (x, S).

5.2 Solution

The policy functions are obtained by approximating f with a linear function. The policy functions are then linear. In general the system can be represented as:

$$f\left(x_t, d_t, s_t, x_{t+1}, d_{t+1}, s_{t+1}\right) \approx A_1 \begin{bmatrix} \hat{x}_t \\ \hat{d}_t \end{bmatrix} + A_2 E \begin{bmatrix} \hat{x}_{t+1} \\ \hat{d}_{t+1} \end{bmatrix} + Z_1 \hat{S}_t + Z_2 E\left[\hat{S}_{t+1}\right] \tag{5.1}$$

Where hatted variables represent deviations from the steady state. The objective is to find laws of motion of the form:

$$\begin{aligned} \hat{x}_{t+1} &= A\hat{x}_t + B\hat{S}_t \\ \hat{d}_t &= C\hat{x}_t + D\hat{S}_t \end{aligned}$$

given that:

$$\hat{S}_{t+1} = P\hat{S}_t + \varepsilon_{t+1}$$

By certainty equivalence matrices A and C can be obtained from solving the non-stochastic model where:

$$A_1 \begin{bmatrix} \hat{x}_t \\ \hat{d}_t \end{bmatrix} + A_2 \begin{bmatrix} \hat{x}_{t+1} \\ \hat{d}_{t+1} \end{bmatrix} = 0$$

(so that the first order conditions are equal to zero).

1. If A_2 is invertible let $\mathcal{A} = -A_2^{-1}A_1$ and then the system is:

$$\begin{bmatrix} \hat{x}_{t+1} \\ \hat{d}_{t+1} \end{bmatrix} = -A_2^{-1}A_1 \begin{bmatrix} \hat{x}_t \\ \hat{d}_t \end{bmatrix} = \mathcal{A} \begin{bmatrix} \hat{x}_t \\ \hat{d}_t \end{bmatrix}$$

   Consider the Eigen-decomposition of $\mathcal{A} = V\Omega V^{-1}$ where the first n_x eigenvalues of $\mathcal{A}$ are inside the unit circle (these correspond to the variables in x) the product can be expressed as:

$$\mathcal{A} = \begin{bmatrix} v_x & v_{xd} \\ v_{dx} & v_d \end{bmatrix} \begin{bmatrix} \Omega_x & 0 \\ 0 & \Omega_d \end{bmatrix} \begin{bmatrix} v_x & v_{xd} \\ v_{dx} & v_d \end{bmatrix}^{-1}$$

   Matrices A and C are then:

$$A = v_x \Omega_x v_x^{-1} \qquad C = v_{dx} v_x^{-1}$$

2. If A_2 is not invertible then generate $\mathcal{A} = V\Omega V^{-1}$ where Ω are the generalized eigenvalues of A_1 and A_2. A and C are defined as before.

Knowing A and C its possible to find B and D by replacing on the FOC. Note that $E\left[\hat{S}_{t+1}\right] = P\hat{S}_t$

$$
\begin{aligned}
A_1 \begin{bmatrix} \hat{x}_t \\ \hat{d}_t \end{bmatrix} + A_2 E_t \begin{bmatrix} \hat{x}_{t+1} \\ \hat{d}_{t+1} \end{bmatrix} + Z_1 \hat{S}_t + Z_2 E_t \left[\hat{S}_{t+1}\right] &= 0 \\
A_1 \begin{bmatrix} \hat{x}_t \\ C\hat{x}_t + D\hat{S}_t \end{bmatrix} + A_2 E_t \begin{bmatrix} \hat{x}_{t+1} \\ C\hat{x}_{t+1} + D\hat{S}_{t+1} \end{bmatrix} + \left(Z_1 + Z_2 P\right) \hat{S}_t &= 0 \\
A_1 \begin{bmatrix} \hat{x}_t \\ C\hat{x}_t + D\hat{S}_t \end{bmatrix} + A_2 \begin{bmatrix} A\hat{x}_t + B\hat{S}_t \\ CA\hat{x}_t + \left(CB + DP\right) \hat{S}_t \end{bmatrix} + \left(Z_1 + Z_2 P\right) \hat{S}_t &= 0
\end{aligned}
$$

Letting $A_1 = [A_{1x}\, A_{1d}]$ and $A_2 = [A_{2x}\, A_{2d}]$ one has:

$$
\begin{aligned}
A_{1x}x_t + A_{1d}\left(C\hat{x}_t + D\hat{S}_t\right) + A_{2x}\left(A\hat{x}_t + B\hat{S}_t\right) + A_{2d}\left(CA\hat{x}_t + \left(CB + DP\right)\hat{S}_t\right) + \left(Z_1 + Z_2 P\right)\hat{S}_t &= 0 \\
\left(A_{1x} + A_{1d}C + A_{2x}A + A_{2d}CA\right)\hat{x}_t + \left(A_{1d}D + A_{2x}B + A_{2d}CB + A_{2d}DP + Z_1 + Z_2 P\right)\hat{S}_t &= 0 \\
\left(A_{1x} + A_{1d}C + A_{2x}A + A_{2d}CA\right)\hat{x}_t + \left(A_{1d}D + A_{2d}DP + \left(A_{2x} + A_{2d}C\right)B + Z_1 + Z_2 P\right)\hat{S}_t &= 0
\end{aligned}
$$

At this point it can be checked that:

$$
A_{1x} + A_{1d}C + A_{2x}A + A_{2d}CA = 0
$$

And then B and D are obtained such that:

$$
A_{1d}D + A_{2d}DP + \left(A_{2x} + A_{2d}C\right)B + Z_1 + Z_2 P = 0_{(n_x+n_d)\times n_s}
$$

Vectorizing:

$$
\begin{aligned}
\text{vec}\left(A_{1d}D\right) + \text{vec}\left(A_{2d}DP\right) + \text{vec}\left(\left(A_{2x} + A_{2d}C\right)B\right) + \text{vec}\left(Z_1 + Z_2 P\right) &= 0 \\
\left(I_{n_s} \otimes A_{1d} + P^{'} \otimes A_{2d}\right)\text{vec}\left(D\right) + \left(I_{n_s} \otimes \left(A_{2x} + A_{2d}C\right)\right)\text{vec}\left(B\right) + \text{vec}\left(Z_1 + Z_2 P\right) &= 0
\end{aligned}
$$

The system of equations can be stacked to give:

$$
\begin{bmatrix} \left(I_{n_s} \otimes \left(A_{2x} + A_{2d}C\right)\right) & \left(I_{n_s} \otimes A_{1d} + P^{'} \otimes A_{2d}\right) \end{bmatrix} \begin{bmatrix} \text{vec}\left(B\right) \\ \text{vec}\left(D\right) \end{bmatrix} = -\text{vec}\left(Z_1 + Z_2 P\right)
$$

$$
\begin{bmatrix} \text{vec}\left(B\right) \\ \text{vec}\left(D\right) \end{bmatrix} = -\begin{bmatrix} \left(I_{n_s} \otimes \left(A_{2x} + A_{2d}C\right)\right) & \left(I_{n_s} \otimes A_{1d} + P^{'} \otimes A_{2d}\right) \end{bmatrix}^{-1} \text{vec}\left(Z_1 + Z_2 P\right)
$$

With this the matrices A, B, C and D are known and the policy functions are fully characterized.

Part II

Economies with distortions

The following sections study the a prototype economy with distortions as the one introduced in Chari, Kehoe & McGrattan (2008). Because of the distortions the methods presented in the part (I) must be modified. In particular its no longer possible to replace the agents in the economy by a central planner as in the stochastic growth model of section 2. Agents in the prototype model must take prices and aggregate quantities as given when taking decisions.

Section 6 presents the model, its first order conditions and steady state. Section 7 makes a short reference to the dynamic programming problem that solves the model. Section 8 solves for a linear-quadratic approximation of the problem, unlike the LQ problem already solved only the policy functions are obtained. Finally, Section 9 modifies the method presented in section 5 to solve for the distorted model.

6 Prototype model

Consider the following growth model where variables are already detrended:

$$\begin{aligned} \max_{\{k_{t+1},c_t,x_t,h_t\}} \quad & E\left[\sum_{t=0}^{\infty}\left(\left(1+\gamma_n\right)\beta\right)^t\left(\log c_t+\psi\log\left(1-h_t\right)\right)\right] \\ \text{s.t.} \quad & 0=r_tk_t+\left(1-\tau_{nt}\right)w_th_t+T_t-c_t-\left(1+\tau_{xt}\right)x_t \\ & 0=\left(1-\delta\right)k_t+x_t-\left(1+\gamma_n\right)\left(1+\gamma_z\right)k_{t+1} \end{aligned}$$

Where $S_t = P_0 + PS_{t-1} + \Sigma\epsilon_t$ and ϵ is distributed iid $N(0, I)$. The firm's technology is $Y_t = K_t^{\alpha}\left(z_tH_t\right)^{1-\alpha}$. The resource constraint of the economy is $Y_t = C_t + X_t + G_t$. Upper case variables represent per-capita aggregates. $S_t = \{\ln z_t, \tau_{xt}, \tau_{nt}, \ln G_t\}$ and $T_t = \tau_{xt}X_t + \tau_{nt}w_tH_t - G_t$.

6.1 FOC

The FOC of an individual household are:

$$\begin{aligned} \frac{1}{c_t} &= \lambda_t \\ \frac{\psi}{1-h_t} &= \lambda_t\left(1-\tau_{nt}\right)w_t \\ \left(1+\tau_{xt}\right)\left(1+\gamma_n\right)\left(1+\gamma_z\right)\lambda_t &= \beta\left(1+\gamma_n\right)\lambda_{t+1}\left(r_{t+1}+\left(1+\tau_{xt+1}\right)\left(1-\delta\right)\right) \end{aligned}$$

From the FOC of the firm one gets:

$$r_t=\alpha\left(\frac{z_tH_t}{K_t}\right)^{1-\alpha} \qquad w_t=\left(1-\alpha\right)z_t\left(\frac{z_tH_t}{K_t}\right)^{-\alpha}$$

Since all households are identical it follows that, in equilibrium:

$$K_t=k_t \quad H_t=h_t \quad X_t=x_t$$

Then the resource constraint of the economy is:

$$k_t^{\alpha}\left(z_th_t\right)^{1-\alpha}+\left(1-\delta\right)k_t=c_t+G_t+\left(1+\gamma_n\right)\left(1+\gamma_z\right)k_{t+1}$$

The exogenous processes follow:

$$S_t = P_0 + PS_{t-1} + \Sigma\epsilon_t$$

The FOC are then:

$$0 = \frac{(1-\tau_{nt})(1-\alpha) z_t \left(\frac{z_t h_t}{k_t}\right)^{-\alpha}}{c_t} - \frac{\psi}{1-h_t} \tag{6.1}$$

$$0 = \frac{\beta}{c_{t+1}} \left(\alpha \left(\frac{z_{t+1} h_{t+1}}{k_{t+1}} \right)^{1-\alpha} + (1+\tau_{xt+1})(1-\delta) \right) - \frac{(1+\tau_{xt})(1+\gamma_z)}{c_t} \tag{6.2}$$

$$0 = k_t^{\alpha} (z_t h_t)^{1-\alpha} + (1-\delta) k_t - c_t - G_t - (1+\gamma_n)(1+\gamma_z) k_{t+1} \tag{6.3}$$

6.2 (Non-Stochastic) Steady state

The (non-stochastic) steady state of the model can be obtained from the conditions above. First the exogenous processes satisfy:

$$S_{ss} = (I-P)^{-1} P_0$$

From the resource constraint consumption satisfies:

$$c_{ss} = k_{ss}^{\alpha} (z_{ss} h_{ss})^{1-\alpha} + (1-\delta-(1+\gamma_n)(1+\gamma_z)) k_{ss} - G_{ss}$$

From the FOC of the household:

$$\begin{aligned}
(1+\tau_{xss})(1+\gamma_z) &= \beta \left(r_{ss} + (1+\tau_{xss})(1-\delta) \right) \\
(1+\tau_{xss}) \left(\frac{(1+\gamma_z)}{\beta} - (1-\delta) \right) &= \alpha \left(\frac{z_{ss} h_{ss}}{k_{ss}} \right)^{1-\alpha} \\
\left(\left(\frac{1+\tau_{xss}}{\alpha} \right) \left(\frac{(1+\gamma_z)}{\beta} - (1-\delta) \right) \right)^{\frac{1}{1-\alpha}} &= \frac{z_{ss} h_{ss}}{k_{ss}} \\
\Lambda_1 &= \frac{z_{ss} h_{ss}}{k_{ss}}
\end{aligned}$$

This implies:

$$w_{ss} = (1-\alpha) z_{ss} \left(\frac{z_{ss} H_{ss}}{K_{ss}} \right)^{-\alpha} = (1-\alpha) z_{ss} \Lambda_1^{-\alpha}$$

One can set the value of G_{ss} so that is some given percentage of the output in steady state, that way: $G_{ss} = \phi_g Y_{ss} = \phi_g k_{ss}^{\alpha} (z_{ss} h_{ss})^{1-\alpha}$.

From the resource constraint consumption satisfies:

$$\begin{aligned}
c_{ss} &= k_{ss}^{\alpha} (z_{ss} h_{ss})^{1-\alpha} + (1-\delta-(1+\gamma_n)(1+\gamma_z)) k_{ss} - G_{ss} \\
&= (1-\phi_g) k_{ss}^{\alpha} (z_{ss} h_{ss})^{1-\alpha} + (1-\delta-(1+\gamma_n)(1+\gamma_z)) k_{ss} \\
&= \left((1-\phi_g) \Lambda_1^{1-\alpha} + (1-\delta-(1+\gamma_n)(1+\gamma_z)) \right) k_{ss} \\
&= \Lambda_2 k_{ss}
\end{aligned}$$

Replacing one gets:

$$\begin{aligned}
\frac{\psi c_{ss}}{1-h_{ss}} &= (1-\tau_{nss}) w_{ss} \\
\psi \Lambda_2 k_{ss} &= (1-\tau_{nss}) w_{ss} (1-h_{ss}) \\
\psi \Lambda_2 k_{ss} + (1-\tau_{nss}) w_{ss} h_{ss} &= (1-\tau_{nss}) w_{ss} \\
\left(\psi \Lambda_2 + \frac{(1-\tilde{\tau}_{nss}) w_{ss} \Lambda_1}{z_{ss}} \right) k_{ss} &= (1-\tau_{nss}) w_{ss}
\end{aligned}$$

$$k_{ss} = \left[\psi \Lambda_2 + \frac{(1-\tau_{nss}) w_{ss} \Lambda_1}{z_{ss}} \right]^{-1} (1-\tau_{nss}) w_{ss}$$

6.3 Additional variables

From the solution to the problem above one can also get dividends, firm's accounting profits and stock valuations.

Dividends are defined as firm's profits that are not used for investment:

$$d_t = K_t^{\alpha} \left(z_t H_t\right)^{1-\alpha} - w_t H_t - \left(1 + \tau_{xt}\right) X_t$$

Note that it is assumed that firms pay wages to households who then pay taxes on them to the government, while firms are the ones investing in new capital and pay investment taxes.

Accounting profits are given by dividends plus capital replacement:

$$\mathrm{Pr}_t = d_t + k_{t+1} - k_t$$

Finally stock valuations are obtained from Tobin's Q. In the model

$$Q_t = \left(1 + \tau_{xt}\right)$$

and

$$Q_t = \frac{v_t}{K_t}$$

Then

$$v_t = \left(1 + \tau_{xt}\right) K_t$$

7 Recursive competitive equilibrium

The full non-linear solution of the problem takes the form of a recursive competitive equilibria.

An RCE is a set formed by a value function V, policy functions g_k and g_h and transition functions G_k and G_h such that:

1. The value function V solves for the following functional equation:

$$\begin{aligned}
V(k,S,K) &= \max_{k',h}\left\{\log c+\psi\log(1-h)+(1+\gamma_n)\beta E\left[V\left(k',S',K'\right)|S\right]\right\}\\
\text{s.t.} \quad & c = rk+(1-\tau_n)wh+\Upsilon-(1+\tau_x)x\\
& x=(1+\gamma_n)(1+\gamma_z)k'-(1-\delta)k\\
& r=\alpha\left(\frac{zH}{K}\right)^{1-\alpha}\\
& w=(1-\alpha)z\left(\frac{zH}{K}\right)^{-\alpha}\\
& \Upsilon=\tau_x\left((1+\gamma_n)(1+\gamma_z)K'-(1-\delta)K\right)+\tau_n wH-g\\
& S'=P_0+P_1S+\Sigma\epsilon\\
& H=G_h(S,K)\\
& K'=G_k(S,K)
\end{aligned}$$

2. The policy functions g_k and g_h are such that:

$$V(k,S,K)=\log c^{\star}+\psi\log(1-g_h(k,S,K))+(1+\gamma_n)\beta E\left[V\left(g_k(k,S,K),S',K'\right)|S\right]$$

   Where $c^{\star}$ evaluates all definitions at $h=g_h(k,S,K)$ and $k'=g_k(k,S,K)$.

3. The aggregate states move according to:

4.

$$\begin{aligned}
G_k(S,K) &= g_k(K,S,K)\\
G_h(S,K) &= g_h(K,S,K)
\end{aligned}$$

To solve the RCE with a recursive procedure note that the problem that V solves is indexed by G_k and G_h, then one can pick a G_k^0 and a G_h^0 (arbitrarily) and solve for V^0, g_k^0, g_h^0 given the guess for aggregate policy functions. One can update the guess by the rule in (3) so that:

$$G_k^{n+1}(S,K)=g_k^n(K,S,K) \qquad G_h^{n+1}(S,K)=g_h^n(K,S,K)$$

And continue until convergence is achieved between the policy functions.

8 Linear-Quadratic approximation - Distortions

Let $x_{1,t} = \ln k_t$, $x_{2,t} = s_t = [\ln z_t, \tau_{xt}, \tau_{nt}, \ln g_t]^{'}$, $x_{3,t} = [\ln K_t, \ln K_{t+1}, H_t]^{'}$ and $x_t = \left[x_{1,t}, x_{2,t}^{'}, x_{3,t}^{'}\right]^{'}$ be the states of the problem. x_{1t} if formed by endogenous individual states (in this case only capital), $x_{2,t}$ is formed by exogenous states contained in vector s_t, and $x_{3,t}$ contains the aggregate states. Let $u_t = [\ln k_{t+1}, h_t]^{'}$ be the vector of household decisions.

The one-period return function of the problem is given by:

$$r(x_t, u_t) = \log\left(r_t e^{\ln k_t} + (1-\tau_{nt}) w_t h_t + \Upsilon_t - (1+\tau_{xt}) x_t\right) + \psi \log(1-h_t)$$

Where:

$$r_t = \alpha \left(\frac{z_t H_t}{e^{\ln K_t}}\right)^{1-\alpha} \qquad w_t = (1-\alpha) z_t \left(\frac{z_t H_t}{e^{\ln K_t}}\right)^{-\alpha}$$

$$x_t = (1+\gamma_n)(1+\gamma_z) e^{\ln k_{t+1}} - (1-\delta) e^{\ln k_t}$$

$$\Upsilon_t = \tau_{xt}\left((1+\gamma_n)(1+\gamma_z) e^{\ln K_{t+1}} - (1-\delta) e^{\ln K_t}\right) + \tau_{nt} w_t H_t - e^{\ln g_t}$$

Note that $r_t, w_t, x_t, \Upsilon_t$ are only definitions and don't come into the problem as variables, they are completely characterized by the states and controls.

A second order approximation around the steady state takes the following form (where $\hat{x}_{i,t} = x_{i,t} - x_{i,ss}$ and $\hat{u}_t = u_t - u_{ss}$):

$$r(x_t, u_t) \approx \frac{1}{2}\begin{bmatrix} 1 & \tilde{x}_{1,t} & \hat{x}_{2,t} & \hat{x}_{3,t} & \hat{u}_t \end{bmatrix} \begin{bmatrix} 2r & J_r^{'} \\ J_r & H_r \end{bmatrix} \begin{bmatrix} 1 \\ \hat{x}_{1,t} \\ \hat{x}_{2,t} \\ \hat{x}_{3,t} \\ \hat{u}_t \end{bmatrix} = \begin{bmatrix} \hat{x}_t^{'} & \hat{u}_t^{'} \end{bmatrix} \begin{bmatrix} Q_{9\times 9} & W_{9\times 2} \\ W_{2\times 9}^{'} & R_{2\times 2} \end{bmatrix} \begin{bmatrix} \hat{x}_t \\ \hat{u}_t \end{bmatrix}$$

$$r(x_t, u_t) \approx \hat{x}_t^{'} Q \hat{x}_t + \hat{u}_t^{'} R \hat{u}_t + \hat{x}_t^{'} W \hat{u}_t$$

Where $\hat{x}_t = [1, \hat{x}_{1,t}, \hat{x}_{2,t}, \hat{x}_{3,t}]^{'}$.

Law of motion for the states is given by:

$$\hat{x}_{t+1} = A\hat{x}_t + B\hat{u}_t + C\epsilon_{t+1}$$

$$\begin{bmatrix} 1 \\ \hat{x}_{1,t+1} \\ \hat{x}_{2,t+1} \\ \hat{x}_{3,t+1} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & A_{11} & A_{12} & A_{13} \\ 0 & 0_{4\times 1} & A_{22} & A_{23} \\ 0 & 0_{3\times 1} & A_{32} & A_{33} \end{bmatrix} \begin{bmatrix} 1 \\ \hat{x}_{1,t} \\ \hat{x}_{2,t} \\ \hat{x}_{3,t} \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0_{4\times 1} & 0_{4\times 1} \\ 0_{3\times 1} & 0_{3\times 1} \end{bmatrix} \begin{bmatrix} \tilde{u}_{1,t} \\ \tilde{u}_{2,t} \end{bmatrix} + \begin{bmatrix} 0_{1\times 4} \\ 0_{1\times 4} \\ \Sigma_{4\times 4} \\ 0_{3\times 4} \end{bmatrix} \epsilon_{t+1}$$

Note that in this case $A_{11} = 0_{1\times 1}$, $A_{12} = 0_{1\times 4}$ and $A_{13} = 0_{1\times 3}$ since next period's capital is given directly by the decision made today. It also holds that $A_{22} = P_1$, since variables are expressed in deviations from the SS there is no constant term P_0, also $A_{23} = 0_{4\times 3}$. Matrices A_{32} and A_{33} are unknown but they are not needed in what follows.

The problem is then:

$$\max_{\{\hat{u}_t, \hat{x}_{1,t+1}\}} E \sum \hat{\beta}^t \left(\hat{x}_t^{'} Q \hat{x}_t + \hat{u}_t^{'} R \hat{u}_t + \hat{x}_t^{'} W \hat{u}_t\right) \qquad \text{s.t. } \hat{x}_{t+1} = A\hat{x}_t + B\hat{u}_t + C\epsilon_{t+1}$$

Next one can re-define variables and matrices to get rid of $\hat{\beta}^t$ and the cross product:

$$
\begin{aligned}
\bar{x}_t &= \hat{\beta}^{t/2}\hat{x}_t \\
\overline{u}_t &= \hat{\beta}^{t/2}\left(\hat{u}_t + R^{-1}W^{'}\hat{x}_t\right) \\
\overline{A} &= \sqrt{\hat{\beta}}\left(A - BR^{-1}W^{'}\right) \\
\overline{B} &= \sqrt{\hat{\beta}}B \\
\overline{Q} &= Q - WR^{-1}W^{'}
\end{aligned}
$$

The problem is then:

$$
\max_{\{\overline{u}_t, \overline{x}_{1,t+1}\}} E\sum\left(\overline{x}_t^{'}\overline{Q}\overline{x}_t + \overline{u}_t^{'}R\overline{u}_t\right) \qquad \text{s.t. } \overline{x}_{t+1} = \overline{A}\overline{x}_t + \overline{B}\overline{u}_t + \overline{C}\overline{\epsilon}_{t+1}
$$

For convenience let $\overline{y}_t = \left[1, \overline{x}_{1,t}, \overline{x}_{2,t}^{'}\right]^{'}$ and $\overline{z}_t = \overline{x}_{3,t}$ and name the following matrices $\overline{A}_y$, $\overline{A}_z$ and $\overline{B}_y$ so that the law of motion is:

$$
\begin{aligned}
\overline{x}_{t+1} &= \overline{A}\overline{x}_t + \overline{B}\overline{u}_t + \overline{C}\overline{\epsilon}_{t+1} \\
\begin{bmatrix} \overline{y}_{t+1} \\ \overline{z}_{t+1} \end{bmatrix} &= \begin{bmatrix} \overline{A}_y & \overline{A}_z \\ \begin{bmatrix} 0 & \overline{A}_{32} \end{bmatrix} & \overline{A}_{33} \end{bmatrix}\begin{bmatrix} \overline{y}_t \\ \overline{z}_t \end{bmatrix} + \begin{bmatrix} \overline{B}_y \\ 0_{3\times 2} \end{bmatrix}\overline{u}_t + \overline{C}\overline{\epsilon}_{t+1}
\end{aligned}
$$

There are two methods for solving this problem, one can use the dynamic programming representation of it and guess the form of the value function and then obtain the coefficients by solving a Ricatti equation recursively, as in section 4.1, or one can using the first order conditions and the Vaughan approach, as in section 4.2. Solving for the Ricatti equation requires to know $\overline{A}_{32}$ and $\overline{A}_{33}$, then only the Vaughan approach is pursued.

8.1 Vaughan's method to LQ method - Distortions

Recall the problem and disregard the stochastic part:

$$
\max_{\{\overline{u}_t, \overline{x}_{t+1}\}} \sum\left(\overline{x}_t^{'}\overline{Q}\overline{x}_t + \overline{u}_t^{'}R\overline{u}_t\right) \qquad \text{s.t. } \overline{x}_{t+1} = \overline{A}\overline{x}_t + \overline{B}\overline{u}_t
$$

Suppose that the whole sequence for aggregate variables $\overline{z}_t$ is known. Noting matrix $\overline{Q}$ as:

$$
\overline{Q} = \begin{bmatrix} \overline{Q}_y & \overline{Q}_z \\ \overline{Q}_z & \overline{Q}_{zz} \end{bmatrix}
$$

the problem takes the form:

$$
\max_{\{\overline{u}_t, \overline{y}_{t+1}\}} \sum\left(\overline{y}_t^{'}\overline{Q}_y\overline{y}_t + 2\overline{y}_t^{'}\overline{Q}_z\overline{z}_t + \overline{z}_t^{'}\overline{Q}_{zz}\overline{z}_t + \overline{u}_t^{'}R\overline{u}_t\right) \qquad \text{s.t. } \overline{y}_{t+1} = \overline{A}_y\overline{y}_t + \overline{A}_z\overline{z}_t + \overline{B}_y\overline{u}_t
$$

Letting $2\lambda_{t+1}$ be the multiplier on the restriction for $\overline{y}_{t+1}$ one gets:

$$
\begin{aligned}
2R\overline{u}_t &= -2\overline{B}_y^{'}\lambda_{t+1} && (8.1) \\
2\overline{Q}_y\overline{y}_{t+1} + 2\overline{Q}_z\overline{z}_{t+1} &= 2\lambda_{t+1} - 2\overline{A}_y^{'}\lambda_{t+2} && (8.2) \\
\overline{y}_{t+1} &= \overline{A}_y\overline{y}_t + \overline{A}_z\overline{z}_t + \overline{B}_y\overline{u}_t && (8.3)
\end{aligned}
$$

From the first equation

$$
\overline{u}_t = -R^{-1}\overline{B}_y^{'}\lambda_{t+1} \qquad (8.4)
$$

This equation will be used to obtain the policy function of the problem. First its necessary to eliminate $\overline{z}$ form the system, then one can express λ as a function of the endogenous states ($\overline{y}$), with this the policy function will be known.

Using (8.4) and lagging equation (9) one gets (solving for $\overline{y}_t$ from the third equation and λ_t from the second):

$$\begin{aligned} \overline{A}_y \overline{y}_t &= \overline{y}_{t+1} - \overline{A}_z \overline{z}_t + \overline{B}_y R^{-1} \overline{B}_y^{'} \lambda_{t+1} \\ \lambda_t &= \overline{Q}_y \overline{y}_t + \overline{Q}_z \overline{z}_t + \overline{A}_y^{'} \lambda_{t+1} \end{aligned}$$

At the same same time it is known from market clearing conditions that in equilibrium:

$$K_t = k_t \qquad K_{t+1} = k_{t+1} \qquad H_t = h_t$$

This allows to express a law of motion for $\hat{z}$ as:

$$\begin{aligned} \hat{z}_t &= \Theta \hat{y}_t + \Psi \hat{u}_t \\ \begin{bmatrix} \widehat{\ln K_t} \\ \widehat{\ln K_{t+1}} \\ \hat{H}_t \end{bmatrix} &= \begin{bmatrix} \begin{bmatrix} 0 & 1 & 0_{1\times 4} \end{bmatrix} \\ 0_{1\times 6} \\ 0_{1\times 6} \end{bmatrix} \hat{y}_t + \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \hat{u}_t \end{aligned}$$

This condition has to be mapped to the modified variables. Recalling that $\overline{u}_t = \hat{\beta}^{t/2} \hat{u}_t + R^{-1} W^{'} \overline{x}_t$ and naming

$$R^{-1} W^{'} = \begin{bmatrix} \Phi_y & \Phi_z \end{bmatrix}$$

one gets:

$$\begin{aligned} \hat{z}_t &= \Theta \hat{y}_t + \Psi \hat{u}_t \\ \overline{z}_t &= \Theta \overline{y}_t + \Psi \beta^{t/2} \hat{u}_t \\ \overline{z}_t &= \Theta \overline{y}_t + \Psi \left(\overline{u}_t - \Phi_y \overline{y}_t - \Phi_z \overline{z}_t \right) \\ \overline{z}_t &= \left(I_2 + \Psi \Phi_z \right)^{-1} \left(\Theta - \Psi \Phi_y \right) \overline{y}_t + \left(I_2 + \Psi \Phi_z \right)^{-1} \Psi \overline{u}_t \\ \overline{z}_t &= \overline{\Theta} \overline{y}_t + \overline{\Psi} \overline{u}_t \end{aligned}$$

Replacing for $\overline{u}$:

$$\begin{aligned} \overline{z}_t &= \left(I_2 + \Psi \Phi_z \right)^{-1} \left(\Theta - \Psi \Phi_y \right) \overline{y}_t - \left(I_2 + \Psi \Phi_z \right)^{-1} \Psi R^{-1} \overline{B}_y^{'} \lambda_{t+1} \\ \overline{z}_t &= \overline{\Theta} \overline{y}_t - \overline{\Psi} \lambda_{t+1} \end{aligned}$$

One can then replace for $\overline{z}$ in the system of first order conditions:

$$\begin{aligned} \overline{A}_y \overline{y}_t &= \overline{y}_{t+1} - \overline{A}_z \left(\overline{\Theta} \overline{y}_t - \overline{\Psi} \lambda_{t+1} \right) + \overline{B}_y R^{-1} \overline{B}_y^{'} \lambda_{t+1} \\ \lambda_t &= \overline{Q}_y \overline{y}_t + \overline{Q}_z \left(\overline{\Theta} \overline{y}_t - \overline{\Psi} \lambda_{t+1} \right) + \overline{A}_y^{'} \lambda_{t+1} \end{aligned}$$

Joining terms:

$$\begin{aligned} \left(\overline{A}_y + \overline{A}_z \overline{\Theta} \right) \overline{y}_t &= \overline{y}_{t+1} + \left(\overline{A}_z \overline{\Psi} + \overline{B}_y R^{-1} \overline{B}_y^{'} \right) \lambda_{t+1} \\ \lambda_t &= \left(\overline{Q}_y + \overline{Q}_z \overline{\Theta} \right) \overline{y}_t + \left(\overline{A}_y^{'} - \overline{Q}_z \overline{\Psi} \right) \lambda_{t+1} \end{aligned}$$

Let

$$\begin{aligned} \tilde{A} &= \overline{A}_y + \overline{A}_z \overline{\Theta} \\ \tilde{B} &= \overline{A}_z \overline{\Psi} + \overline{B}_y R^{-1} \overline{B}_y^{'} \\ \tilde{Q} &= \overline{Q}_y + \overline{Q}_z \overline{\Theta} \end{aligned}$$

so that the system is:

$$\tilde{A}\overline{y}_t = \overline{y}_{t+1} + \tilde{B}\lambda_{t+1} \tag{8.5}$$

$$\lambda_t = \tilde{Q}\overline{y}_t + \left(\overline{A}_y^{'} - \overline{Q}_z\overline{\Psi}\right)\lambda_{t+1} \tag{8.6}$$

As before there are two cases for getting the dynamical system.

1. If $\tilde{A}$ is invertible the equations are:

$$\begin{aligned} \overline{y}_t &= \tilde{A}^{-1}\overline{y}_{t+1} + \tilde{A}^{-1}\tilde{B}\lambda_{t+1} \\ \lambda_t &= \tilde{Q}\overline{y}_t + \left(\overline{A}_y^{'} - \overline{Q}_z\overline{\Psi}\right)\lambda_{t+1} \end{aligned}$$

   To get the dynamical system one replaces $\overline{y}_t$ from the first equation

$$\begin{aligned} \overline{y}_t &= \tilde{A}^{-1}\overline{y}_{t+1} + \tilde{A}^{-1}\tilde{B}\lambda_{t+1} \\ \lambda_t &= \tilde{Q}\tilde{A}^{-1}\overline{y}_{t+1} + \left(\tilde{Q}\tilde{A}^{-1}\tilde{B} + \overline{A}_y^{'} - \overline{Q}_z\overline{\Psi}\right)\lambda_{t+1} \end{aligned}$$

   In matrix form:

$$\begin{bmatrix} \overline{y}_t \\ \lambda_t \end{bmatrix} = H \begin{bmatrix} \overline{y}_{t+1} \\ \lambda_{t+1} \end{bmatrix} \qquad H = \begin{bmatrix} \tilde{A}^{-1} & \tilde{A}^{-1}\tilde{B} \\ \tilde{Q}\tilde{A}^{-1} & \left(\tilde{Q}\tilde{A}^{-1}\tilde{B} + \overline{A}_y^{'} - \overline{Q}_z\overline{\Psi}\right) \end{bmatrix}$$

2. If $\tilde{A}$ is not invertible then:

$$H_1 \begin{bmatrix} \overline{y}_t \\ \lambda_t \end{bmatrix} = H_2 \begin{bmatrix} \overline{y}_{t+1} \\ \lambda_{t+1} \end{bmatrix}$$

$$H_1 = \begin{bmatrix} \tilde{A} & 0 \\ -\tilde{Q} & I \end{bmatrix} \qquad H_2 = \begin{bmatrix} I & \tilde{B} \\ 0 & \overline{A}_y^{'} - \overline{Q}_z\overline{\Psi} \end{bmatrix}$$

One can then use the eigen-decomposition of matrix H (or the generalized decomposition of matrices H_1 and H_2) to get:

$$H = V\Lambda V^{-1}$$

The eigenvalues of H don't longer come in reciprocal pairs, but there must be as many eigenvalues inside the unit circle as there are states. One can write the following:

$$H = \begin{bmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{bmatrix} \begin{bmatrix} \Lambda_1 & 0 \\ 0 & \Lambda_2 \end{bmatrix} \begin{bmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{bmatrix}^{-1} \tag{8.7}$$

where all eigenvalues in Λ_1 are outside of the unit circle.

Guess that $\lambda_t = S\overline{y}_t$, it is shown in section 8.2 that:

$$S = V_{21}V_{11}^{-1}$$

As before this gives a relation between the controls and the states. Using equation (8.4):

$$\overline{u}_t = -R^{-1}\overline{B}_y^{'}S\overline{y}_{t+1}$$

Then, from equation (8.5) one gets:

$$\begin{aligned} \tilde{A}\overline{y}_t &= \overline{y}_{t+1} + \tilde{B}\lambda_{t+1} \\ \tilde{A}\overline{y}_t &= \overline{y}_{t+1} + \tilde{B}S\overline{y}_{t+1} \\ \left(I + \tilde{B}S\right)^{-1}\tilde{A}\overline{y}_t &= \overline{y}_{t+1} \end{aligned}$$

Joining results one obtains the policy function:

$$\overline{u}_t = -R^{-1}\overline{B}_y^{'} S \left(I + \tilde{B}S\right)^{-1} \tilde{A}\overline{y}_t \tag{8.8}$$

Let $\overline{F} = R^{-1}\overline{B}_y^{'} S \left(I + \tilde{B}S\right)^{-1} \tilde{A}$.

Then for the variables in level we have:

$$\begin{aligned}
\overline{u}_t &= -\overline{F}\overline{y}_t \\
\hat{u}_t + R^{-1}W^{'}\hat{x}_t &= -\overline{F}\hat{y}_t \\
\hat{u}_t + \left(\Phi_y \hat{y}_t + \Phi_z \hat{z}_t\right) &= -\overline{F}\hat{y}_t \\
\hat{u}_t + \left(\Phi_y \hat{y}_t + \Phi\left(\Theta\hat{y}_t + \Psi\hat{u}_t\right)\right) &= -\overline{F}\hat{y}_t \\
\hat{u}_t + \left(\left(\Phi_y + \Phi_z\Theta\right)\hat{y}_t + \Phi_z\Psi\hat{u}_t\right) &= -\overline{F}\hat{y}_t \\
\left(I + \Phi_z\Psi\right)\hat{u}_t + \left(\Phi_y + \Phi_z\Theta\right)\hat{y}_t &= -\overline{F}\hat{y}_t \\
\left(I - R^{-1}W^{'}\Phi_z\Psi\right)\hat{u}_t &= -\left(\overline{F} + \Phi_y + \Phi_z\Theta\right)\hat{y}_t \\
\hat{u}_t &= -\left(I + \Phi_z\Psi\right)^{-1}\left(\overline{F} + \Phi_y + \Phi_z\Theta\right)\hat{y}_t
\end{aligned}$$

Which gives the transition for the original variable $\hat{u}_t = -F\hat{y}_t$ where:

$$F = \left(I + \Phi_z\Psi\right)^{-1}\left(\overline{F} + \Phi_y + \Phi_z\Theta\right) \tag{8.9}$$

The first row of F gives the policy function for capital, the second one the policy function for labor. The exogenous states evolve according to their VAR process.

8.2 Solution to Hamiltonian (by Emily Moschini)

The above problem has the form:

$$\begin{bmatrix} \overline{y}_t \\ \lambda_t \end{bmatrix} = H \begin{bmatrix} \overline{y}_{t+1} \\ \lambda_{t+1} \end{bmatrix}$$

Rearrange the system so that it is moving forward in time.

$$\begin{bmatrix} \overline{y}_{t+1} \\ \lambda_{t+1} \end{bmatrix} = H^{-1} \begin{bmatrix} \overline{y}_t \\ \lambda_t \end{bmatrix}$$

From the eigen-decomposition of the matrix H one has:

$$H^{-1} = \begin{bmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{bmatrix} \begin{bmatrix} \Lambda_1 & 0 \\ 0 & \Lambda_2 \end{bmatrix} \begin{bmatrix} W_{11} & W_{12} \\ W_{21} & W_{22} \end{bmatrix}$$

where the notation has been changed from equation (8.7), Λ_1 represents the eigenvalues inside the unit circle, Λ_2 those outside, and:

$$\begin{bmatrix} W_{11} & W_{12} \\ W_{21} & W_{22} \end{bmatrix} = \begin{bmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{bmatrix}^{-1}$$

Remember that we guess the form of the relationship between the states y_t and co-states λ_t to be $\lambda_t = S\overline{y}_t$. Substitute in this guess to the system above and solve for S, subject to the constraint that it puts 0 weight on the eigenvalues outside the unit circle, Λ_2. **The method below is from "A note on computing competitive equilibria in linear models", by Ellen McGratten, 1992.**

$$\begin{aligned}
\begin{bmatrix} \overline{y}_{t+1} \\ S\overline{y}_{t+1} \end{bmatrix} &= \begin{bmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{bmatrix} \begin{bmatrix} \Lambda_1 & 0 \\ 0 & \Lambda_2 \end{bmatrix} \begin{bmatrix} W_{11} & W_{12} \\ W_{21} & W_{22} \end{bmatrix} \begin{bmatrix} \overline{y}_t \\ S\overline{y}_t \end{bmatrix} \\
\begin{bmatrix} \overline{y}_{t+1} \\ S\overline{y}_{t+1} \end{bmatrix} &= \begin{bmatrix} V_{11}\Lambda_1 W_{11} + V_{12}\Lambda_2 W_{21} & V_{11}\Lambda_1 W_{12} + V_{12}\Lambda_2 W_{22} \\ V_{21}\Lambda_1 W_{11} + V_{22}\Lambda_2 W_{21} & V_{21}\Lambda_1 W_{12} + V_{22}\Lambda_2 W_{22} \end{bmatrix} \begin{bmatrix} \overline{y}_t \\ S\overline{y}_t \end{bmatrix}
\end{aligned}$$

This gives two equations:

$$\begin{aligned}\bar{y}_{t+1} &= (V_{11}\Lambda_1 (W_{11}+W_{12}S) + V_{12}\Lambda_2 (W_{21}+W_{22}S))\,\overline{y}_t \\ \bar{y}_{t+1} &= S^{-1}(V_{21}\Lambda_1 (W_{11}+W_{12}S) + V_{22}\Lambda_2 (W_{21}+W_{22}S))\,\overline{y}_t\end{aligned}$$

Since you want to put zero weight on Λ_2, you can set $S = -W_{22}^{-1}W_{21}$. If the weight on the second term of the two equations is zero, then they become:

$$\begin{aligned}\bar{y}_{t+1} &= (V_{11}\Lambda_1 (W_{11}+W_{12}S))\,\overline{y}_t \\ \bar{y}_{t+1} &= S^{-1}(V_{21}\Lambda_1 (W_{11}+W_{12}S))\,\overline{y}_t\end{aligned}$$

Which means that $S^{-1}V_{21} = V_{11}$ since the two RHS have to be equal. This implies:

$$V_{21}V_{11}^{-1} = S$$

Let's check that this is an equivalent condition on S. Take the system of equations:

$$\begin{aligned}\bar{y}_{t+1} &= \left(V_{11}\Lambda_1 \left(W_{11}+W_{12}V_{21}V_{11}^{-1}\right) + V_{12}\Lambda_2 \left(W_{21}+W_{22}V_{21}V_{11}^{-1}\right)\right)\bar{y}_t \\ \bar{y}_{t+1} &= V_{11}V_{21}^{-1}\left(V_{21}\Lambda_1 \left(W_{11}+W_{12}V_{21}V_{11}^{-1}\right) + V_{22}\Lambda_2 \left(W_{21}+W_{22}V_{21}V_{11}^{-1}\right)\right)\overline{y}_t\end{aligned}$$

$$\begin{aligned}\bar{y}_{t+1} &= \left(V_{11}\Lambda_1 \left(W_{11}+W_{12}V_{21}V_{11}^{-1}\right) + V_{12}\Lambda_2 \left(W_{21}+W_{22}V_{21}V_{11}^{-1}\right)\right)\bar{y}_t \\ \bar{y}_{t+1} &= \left(V_{11}\Lambda_1 \left(W_{11}+W_{12}V_{21}V_{11}^{-1}\right) + V_{11}V_{21}^{-1}V_{22}\Lambda_2 \left(W_{21}+W_{22}V_{21}V_{11}^{-1}\right)\right)\overline{y}_t\end{aligned}$$

$$\begin{aligned}V_{12}\Lambda_2\left(W_{21}+W_{22}V_{21}V_{11}^{-1}\right) - V_{11}V_{21}^{-1}V_{22}\Lambda_2\left(W_{21}+W_{22}V_{21}V_{11}^{-1}\right) &= 0 \\ \left[V_{12}-V_{11}V_{21}^{-1}V_{22}\right]\Lambda_2\left(W_{21}+W_{22}V_{21}V_{11}^{-1}\right) &= 0 \\ W_{21}+W_{22}V_{21}V_{11}^{-1} &= 0 \\ S = V_{21}V_{11}^{-1} &= -W_{22}^{-1}W_{21}\end{aligned}$$

The second-to-last line follows from the next point. Note that the matrix V is invertible, one of its terms is:

$$-V_{22}^{-1}V_{21}V_{11}^{-1} + V_{12}^{-1} = \left(V_{12} - V_{11}V_{21}^{-1}V_{22}\right)^{-1}$$

which means that a necessary condition for V to be invertible is that $V_{12} - V_{11}V_{21}^{-1}V_{22} \neq 0$.

8.3 Vaughan's method for LQ approximation (following Ellen McGrattan's notation)

Recall the problem and disregard the stochastic part:

$$\max_{\{\overline{u}_t,\overline{x}_{t+1}\}} \sum \left(\overline{x}_t'\overline{Q}\overline{x}_t + \overline{u}_t'R\overline{u}_t\right) \qquad \text{s.t. } \overline{x}_{t+1} = \overline{A}\overline{x}_t + \overline{B}\overline{u}_t$$

Suppose that the whole sequence for aggregate variables $\overline{z}_t$ is known. Noting matrix $\overline{Q}$ as:

$$\overline{Q} = \begin{bmatrix} \overline{Q}_y & \overline{Q}_z \\ \overline{Q}_z & \overline{Q}_{zz} \end{bmatrix}$$

the problem takes the form:

$$\max_{\{\overline{u}_t,\overline{y}_{t+1}\}} \sum \left(\overline{y}_t'\overline{Q}_y\overline{y}_t + 2\overline{y}_t'\overline{Q}_z\overline{z}_t + \overline{z}_t'\overline{Q}_{zz}\overline{z}_t + \overline{u}_t'R\overline{u}_t\right) \qquad \text{s.t. } \overline{y}_{t+1} = \overline{A}_y\overline{y}_t + \overline{A}_z\overline{z}_t + \overline{B}_y\overline{u}_t$$

Letting $2\lambda_{t+1}$ be the multiplier on the restriction for $\overline{y}_{t+1}$ one gets:

$$\begin{aligned} 2R\overline{u}_t &= -2\overline{B}_y^{'}\lambda_{t+1} \\ 2\overline{Q}_y\overline{y}_{t+1} + 2\overline{Q}_z\overline{z}_{t+1} &= 2\lambda_{t+1} - 2\overline{A}_y^{'}\lambda_{t+2} \\ \overline{y}_{t+1} &= \overline{A}_y\overline{y}_t + \overline{A}_z\overline{z}_t + \overline{B}_y\overline{u}_t \end{aligned}$$

From the first equation $\overline{u}_t = -R^{-1}\overline{B}_y^{'}\lambda_{t+1}$ and lagging the second equation one period one gets (solving for $\overline{y}_t$ from the third equation and λ_t from the second):

$$\begin{aligned} \overline{A}_y\overline{y}_t &= \overline{y}_{t+1} - \overline{A}_z\overline{z}_t + \overline{B}_yR^{-1}\overline{B}_y^{'}\lambda_{t+1} \\ \lambda_t &= \overline{Q}_y\overline{y}_t + \overline{Q}_z\overline{z}_t + \overline{A}_y^{'}\lambda_{t+1} \end{aligned}$$

At the same same time it is known from market clearing conditions that:

$$\begin{aligned} \hat{z}_t &= \Theta\hat{y}_t + \Psi\hat{u}_t \\ \begin{bmatrix} \widehat{\ln K_t} \\ \widehat{\ln K_{t+1}} \\ \hat{H}_t \end{bmatrix} &= \begin{bmatrix} \begin{bmatrix} 0 & 1 & 0_{1\times 4} \end{bmatrix} \\ 0_{1\times 6} \\ 0_{1\times 6} \end{bmatrix}\hat{y}_t + \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}\hat{u}_t \end{aligned}$$

Letting:

$$\overline{\Theta} = \left(I + \Psi R^{-1}W_z^{'}\right)^{-1}\left(\Theta - \Psi R^{-1}W_y^{'}\right) \qquad \overline{\Psi} = \left(I + \Psi R^{-1}W_z^{'}\right)^{-1}\Psi$$

Where

$$W = \begin{bmatrix} W_y \\ W_z \end{bmatrix}$$

Then:

$$\overline{z}_t = \overline{\Theta}\overline{y}_t + \overline{\Psi}\overline{u}_t$$

One can then replace for $\overline{u}_t$ to get:

$$\overline{z}_t = \overline{\Theta}\overline{y}_t - \overline{\Psi}R^{-1}\overline{B}_y^{'}\lambda_{t+1}$$

And then plug $\overline{z}$ in the system above to get:

$$\begin{aligned} \left(\overline{A}_y + \overline{A}_z\Theta\right)\overline{y}_t &= y_{t+1} + \left(\overline{A}_z\overline{\Psi} + \overline{B}_y\right)R^{-1}\overline{B}_y^{'}\lambda_{t+1} \\ \lambda_t &= \left(\overline{Q}_y + \overline{Q}_z\Theta\right)\overline{y}_t + \left(\overline{A}_y^{'} - \overline{Q}_z\overline{\Psi}R^{-1}\overline{B}_y^{'}\right)\lambda_{t+1} \end{aligned}$$

Replacing $\overline{y}_t$ in the second equation one gets the system:

$$\begin{aligned} \overline{y}_t &= \hat{A}^{-1}y_{t+1} + \hat{A}^{-1}\hat{B}R^{-1}\overline{B}_y^{'}\lambda_{t+1} \\ \lambda_t &= \hat{Q}\hat{A}^{-1}y_{t+1} + \left(\left(\hat{Q}\hat{A}^{-1}\hat{B} - \overline{Q}_z\overline{\Psi}\right)R^{-1}\overline{B}_y^{'} + \overline{A}_y^{'}\right)\lambda_{t+1} \end{aligned}$$

Where :

$$\hat{A} = \overline{A}_y + \overline{A}_z\overline{\Theta} \qquad \hat{Q} = \overline{Q}_y + \overline{Q}_z\overline{\Theta} \qquad \hat{B} = \overline{A}_z\overline{\Psi} + \overline{B}_y$$

And then in matrix form:

$$\begin{bmatrix} \overline{y}_t \\ \lambda_t \end{bmatrix} = H\begin{bmatrix} \overline{y}_{t+1} \\ \lambda_{t+1} \end{bmatrix} \qquad H = \begin{bmatrix} \hat{A}^{-1} & \hat{A}^{-1}\hat{B}R^{-1}\overline{B}_y^{'} \\ \hat{Q}\hat{A}^{-1} & \left(\left(\hat{Q}\hat{A}^{-1}\hat{B} - \overline{Q}_z\overline{\Psi}\right)R^{-1}\overline{B}_y^{'} + \overline{A}_y^{'}\right) \end{bmatrix}$$

One can then use the Eigen-decomposition of matrix H (or the generalized decomposition of matrices H_1 and H_2) to get:

$$H = V\Lambda V^{-1}$$

The eigenvalues of H don't longer come in reciprocal pairs, but there must be as many eigenvalues inside the unit circle as there are states. One can write the following:

$$H = \begin{bmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{bmatrix} \begin{bmatrix} \Lambda_1 & 0 \\ 0 & \Lambda_2 \end{bmatrix} \begin{bmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{bmatrix}^{-1}$$

where all eigenvalues in Λ_1 are outside of the unit circle. It can be showed that $S = V_{21}V_{11}^{-1}$ where S is such that

$$\overline{u}_t = -\left(R + \overline{B}_y^{'} S\hat{B}\right)^{-1} \overline{B}_y^{'} S\hat{A}\overline{y}_t$$

Let $\overline{F} = \left(R + \overline{B}_y^{'} S\hat{B}\right)^{-1} \overline{B}_y^{'} S\hat{A}$.

Then for the variables in level we have:

$$\begin{aligned}
\overline{u}_t &= -\overline{F}\overline{y}_t \\
\hat{u}_t + R^{-1}W^{'}\hat{x}_t &= -\overline{F}\hat{y}_t \\
\hat{u}_t + R^{-1}\left[W_y^{'}\hat{y}_t + W_z^{'}\hat{z}_t\right] &= -\overline{F}\hat{y}_t \\
\hat{u}_t + R^{-1}\left[W_y^{'}\hat{y}_t + W_z^{'}\left(\Theta\hat{y}_t + \Psi\hat{u}_t\right)\right] &= -\overline{F}\hat{y}_t \\
\hat{u}_t + R^{-1}\left[\left(W_y^{'} + W_z^{'}\Theta\right)\hat{y}_t + W_z^{'}\Psi\hat{u}_t\right] &= -\overline{F}\hat{y}_t \\
\left(I + R^{-1}W_z^{'}\Psi\right)\hat{u}_t &= -\left(\overline{F} + R^{-1}\left(W_y^{'} + W_z^{'}\Theta\right)\right)\hat{y}_t \\
\hat{u}_t &= -\left(I + R^{-1}W_z^{'}\Psi\right)^{-1}\left(\overline{F} + R^{-1}\left(W_y^{'} + W_z^{'}\Theta\right)\right)\hat{y}_t
\end{aligned}$$

Which gives the transition for the original variable $\hat{u}_t = -F\hat{y}_t$ where:

$$F = \left(I + R^{-1}W_z^{'}\Psi\right)^{-1}\left(\overline{F} + R^{-1}\left(W_y^{'} + W_z^{'}\Theta\right)\right)$$

The first row of F gives the policy function for capital, the second one the policy function for labor. The exogenous states evolve according to their VAR process.

9 First order conditions - Distortions

An alternative to the problem above is to solve for the system of first order conditions. Consider the original problem's first order conditions. This method is simpler and uses the exact same tools presented in section (5), the only difference lies in the problem the FOC are obtained from. Because of the distortions the household has to take decisions given the prices and aggregate quantities, then, after the FOC are obtained one can use the market clearing conditions to eliminate the aggregate variables.

For the household the conditions are:

$$\begin{aligned} 0 &= \frac{1}{c_t} - \lambda_t \\ 0 &= -\frac{\psi}{1-h_t} + \lambda_t \left(1 - \tau_{nt}\right) w_t \\ 0 &= -\left(1 + \tau_{xt}\right) \gamma \lambda_t + \beta \left(1 + \gamma_n\right) E\left[\lambda_{t+1} \left(r_{t+1} + \left(1 + \tau_{xt+1}\right)(1-\delta)\right)\right] \end{aligned}$$

From the FOC of the firm one gets:

$$r_t = \alpha \left(\frac{z_t H_t}{K_t}\right)^{1-\alpha} \qquad w_t = (1-\alpha) z_t \left(\frac{z_t H_t}{K_t}\right)^{-\alpha}$$

Since all households are identical it follows that, in equilibrium:

$$K_t = k_t \quad H_t = h_t$$

Then the resource constraint of the economy is:

$$0 = k_t^{\alpha} \left(z_t h_t\right)^{1-\alpha} + (1-\delta) k_t - c_t - G_t - \gamma k_{t+1}$$

The exogenous processes follow:

$$S_t = P_0 + P_1 S_{t-1} + \Sigma \epsilon_t$$

In equilibrium one can cease to use the aggregate variables, and the rental rate and wage can be replaced into the household FOC, also λ_t can be eliminated, the the set of conditions is:

$$\begin{aligned} 0 &= -\frac{\psi}{1-h_t} + \frac{1}{c_t} \left(1 - \tau_{nt}\right)(1-\alpha) z_t^{1-\alpha} \left(\frac{h_t}{k_t}\right)^{-\alpha} \\ 0 &= -\left(1 + \tau_{xt}\right) \frac{c_{t+1}}{c_t} + \frac{\beta}{1+\gamma_z} E\left(\alpha \left(\frac{z_{t+1} h_{t+1}}{k_{t+1}}\right)^{1-\alpha} + \left(1 + \tau_{xt+1}\right)(1-\delta)\right) \\ 0 &= k_t^{\alpha} \left(z_t h_t\right)^{1-\alpha} + (1-\delta) k_t - c_t - G_t - \gamma k_{t+1} \\ S_{t+1} &= P_0 + P_1 S_t + \Sigma \epsilon_{t+1} \end{aligned}$$

In the above system the variable c can be eliminated using the third condition, but its not necessary to do so.

This is a system of 7 equations (S has dimension 4) and seven variables $\{c, h, k, S\}$. Define the states $x_t = \ln k_t$, decisions $d_t = \{\ln c_t, h_t\}$ and exogenous states S_t. Then the system has 3 first order difference equations for 3 endogenous variables and it can be represented with a function $f : \mathbb{R}^{16} \to \mathbb{R}^4$ where the domain is all variables $\{\{x_t, d_t, S_t\}, \{x_{t+1}, d_{t+1}, S_{t+1}\}\}$.

One can then linearize around the steady state of the model. This gives:

$$f\left(x_t, d_t, s_t, x_{t+1}, d_{t+1}, s_{t+1}\right) \approx A_1 \begin{bmatrix} \hat{x}_t \\ \hat{d}_t \end{bmatrix} + A_2 E \begin{bmatrix} \hat{x}_{t+1} \\ \hat{d}_{t+1} \end{bmatrix} + Z_1 \hat{S}_t + Z_2 E\left[\hat{S}_{t+1}\right]$$

This is the same function obtained in equation (5.1). The policy functions can be then obtained exactly as in section 5. The only difference was in the construction of the non-linear function f.

10 Simulations

10.1 Using LQ

Using the solution of the LQ problem one gets a law of motion for capital and labor , so that:

$$\begin{bmatrix} \widehat{\ln k_{t+1}} \\ \hat{h}_t \end{bmatrix} = -F \begin{bmatrix} 1 \\ \widehat{\ln k_t} \\ \widehat{s}_t \end{bmatrix}$$

Using the law of motion for $\hat{s}_t$ one has:

$$\hat{s}_{t+1} = P_1 \hat{s}_t + \Sigma \epsilon_{t+1}$$

It is then possible to simulate data for the model given that $\epsilon_t \sim N(0_4, I_{4\times 4})$ following the algorithm below:

1. Set $\widehat{\ln k_0} = 0$.
2. Draw a 4×1 vector of variables ϵ from the standard normal distribution. Call it ϵ_0.
3. Draw a $4 \times T$ matrix of variables ϵ form the standard normal distribution. Call the t^{th} column ϵ_t.
4. Generate $\hat{s}_0 = \Sigma \epsilon_0$.
5. For $t = 0, \ldots, T$ generate:

$$\begin{bmatrix} \widehat{\ln k_{t+1}} \\ \hat{h}_t \\ \hat{s}_{t+1} \end{bmatrix} = \begin{bmatrix} -F \\ \begin{bmatrix} 0_{4\times 2} & P_1 \end{bmatrix} \end{bmatrix} \begin{bmatrix} 1 \\ \widehat{\ln k_t} \\ \widehat{s}_t \end{bmatrix} + \begin{bmatrix} 0_{2\times 4} \\ \Sigma \end{bmatrix} \epsilon_{t+1}$$

This provides time series for the seven variables included in the problem. Then define:

$$k_t = e^{\widehat{\ln k_t} + \ln k_{ss}} \qquad h_t = \hat{h}_t + h_{ss}$$

$$z_t = e^{\widehat{\ln z_t} + \ln z_{ss}} \qquad \tau_{nt} = \hat{\tau}_{nt} + \tau_{n,ss} \qquad \tau_{xt} = \hat{\tau}_{xt} + \tau_{x,ss} \qquad g_t = e^{\widehat{\ln g_t} + \ln g_{ss}}$$

Once one has the time series for the (detrended) levels of the variable one can get any of the other variables of the model using the definitions presented below:

$$\begin{aligned}
x_t &= (1+\gamma_n)(1+\gamma_z) k_{t+1} - (1-\delta) k_t \\
r_t &= \alpha \left(\frac{z_t h_t}{k_t} \right)^{1-\alpha} \\
w_t &= (1-\alpha) z_t \left(\frac{z_t h_t}{k_t} \right)^{-\alpha} \\
\Upsilon_t &= \tau_{xt} \left((1+\gamma_n)(1+\gamma_z) k_{t+1} - (1-\delta) k_t \right) + \tau_{nt} w_t h_t - g_t \\
c_t &= r_t k_t + (1-\tau_{nt}) w_t h_t + \Upsilon_t - (1+\tau_{xt}) x_t \\
d_t &= k_t^{\alpha} (z_t h_t)^{1-\alpha} - w_t h_t - (1+\tau_{xt}) h_t \\
\Pr_t &= d_t + k_{t+1} - k_t \\
v_t &= (1+\tau_{xt}) k_t
\end{aligned}$$

10.2 Using FOC

Using the solution of the FOC one gets a law of motion for capital and labor (and consumption), so that:

$$\widehat{\ln k_{t+1}} = \begin{bmatrix} A & B \end{bmatrix} \begin{bmatrix} \widehat{\ln k_t} \\ \widehat{s}_t \end{bmatrix}$$

and :

$$\begin{bmatrix} \hat{h}_t \\ \widehat{\ln c_t} \end{bmatrix} = \begin{bmatrix} C & D \end{bmatrix} \begin{bmatrix} \widehat{\ln k_t} \\ \widehat{s}_t \end{bmatrix}$$

Using the law of motion for $\hat{s}_t$ one has:

$$\hat{s}_{t+1} = P_1 \hat{s}_t + \Sigma \epsilon_{t+1}$$

It is then possible to simulate data for the model given that $\epsilon_t \sim N(0_4, I_{4\times 4})$ following the algorithm below:

1. Set $\widehat{\ln k_0} = 0$.
2. Draw a 4×1 vector of variables ϵ from the standard normal distribution. Call it ϵ_0.
3. Draw a $4 \times T$ matrix of variables ϵ form the standard normal distribution. Call the t^{th} column ϵ_t.
4. Generate $\hat{s}_0 = \Sigma \epsilon_0$.
5. For $t = 0, \ldots, T$ generate:

$$\begin{bmatrix} \widehat{\ln k_{t+1}} \\ \hat{h}_t \\ \widehat{\ln c_t} \\ \hat{s}_{t+1} \end{bmatrix} = \begin{bmatrix} A_{1\times 1} & B_{1\times 4} \\ C_{2\times 1} & D_{2\times 4} \\ 0_{4\times 1} & P_1 \end{bmatrix} \begin{bmatrix} \widehat{\ln k_t} \\ \widehat{s}_t \end{bmatrix} + \begin{bmatrix} 0_{3\times 4} \\ \Sigma \end{bmatrix} \epsilon_{t+1}$$

This provides time series for the seven variables included in the problem. Then define:

$$k_t = e^{\widehat{\ln k_t} + \ln k_{ss}} \qquad h_t = \hat{h}_t + h_{ss} \qquad c_t = e^{\widehat{\ln c_t} + \ln c_{ss}}$$

$$z_t = e^{\widehat{\ln z_t} + \ln z_{ss}} \qquad \tau_{nt} = \hat{\tau}_{nt} + \tau_{n,ss} \qquad \tau_{xt} = \hat{\tau}_{xt} + \tau_{x,ss} \qquad g_t = e^{\widehat{\ln g_t} + \ln g_{ss}}$$

Once one has the time series for the (detrended) levels of the variable one can get any of the other variables of the model using the definitions presented below:

$$\begin{aligned}
x_t &= (1+\gamma_n)(1+\gamma_z) k_{t+1} - (1-\delta) k_t \\
r_t &= \alpha \left(\frac{z_t h_t}{k_t} \right)^{1-\alpha} \\
w_t &= (1-\alpha) z_t \left(\frac{z_t h_t}{k_t} \right)^{-\alpha} \\
\Upsilon_t &= \tau_{xt} \left((1+\gamma_n)(1+\gamma_z) k_{t+1} - (1-\delta) k_t \right) + \tau_{nt} w_t h_t - g_t \\
d_t &= k_t^{\alpha} (z_t h_t)^{1-\alpha} - w_t h_t - (1+\tau_{xt}) h_t \\
\Pr_t &= d_t + k_{t+1} - k_t \\
v_t &= (1+\tau_{xt}) k_t
\end{aligned}$$

Part III
The BCA procedure

The following sections examine how to establish equivalence results between detailed models and ht prototype model of Section 6. These type of results are necessary to use the BCA procedure. As an example a detailed version of the prototype model is used and equivalence is shown. The detailed version presents news shocks and investment productivity shocks in the spirit of Jaimovich & Rebelo (2009).

Once the equivalence result has been obtained one can use the prototype model (Section 6) to study the data and recover series for the wedges. Contrasting the type of wedges present in the data with the ones implied by the model through the equivalence results is a way to validate the detailed model. This validation does not only rely on the wedged recovered but also on the series implied by each wedge (the wedge series can seem non-important and have a seizable effect over the other variables of the model).

Section 11 presents the detailed prototype model, section 12 the equivalence results with the prototype model of section 6. Then section 13 shows how to recover the wedges from using data on aggregate series and the prototype model. Finally section 14 shows how to simulate the series implied by any given combination of wedges. With this the complete BCA procedure can be implemented.

11 Prototype economy with news shocks and investment productivity

Consider the following growth model where variables are already detrended:

$$\begin{aligned}
\max_{\{k_{t+1},c_t,x_t,h_t\}} \quad & E\left[\sum_{t=0}^{\infty}\left(\left(1+\gamma_n\right)\beta\right)^t\left(\log c_t+\psi\log\left(1-h_t\right)\right)\right] \\
\text{s.t.} \quad & 0=r_tk_t+\left(1-\tilde{\tau}_{nt}\right)w_th_t+\tilde{T}_t-c_t-\left(1+\tilde{\tau}_{xt}\right)\frac{x_t}{z_t^x} \\
& 0=\left(1-\delta\right)k_t+x_t-\left(1+\gamma_n\right)\left(1+\gamma_z\right)k_{t+1}
\end{aligned}$$

Where $\tilde{S}_t=\tilde{P}_0+\tilde{P}\tilde{S}_{t-1}+\tilde{Q}\tilde{\epsilon}_t$ and ϵ is distributed iid $N\left(0,I\right)$. The firm's technology is $Y_t=K_t^{\alpha}\left(\tilde{z}_tH_t\right)^{1-\alpha}$. The resource constraint of the economy is $Y_t=C_t+X_t+\tilde{G}_t$. Upper case variables represent per-capita aggregates. $S_t=\{\ln\tilde{z}_t,\tau_{xt},\tau_{nt},\ln G_t,\ln z_t^x\}$ and $\tilde{T}_t=\tilde{\tau}_{xt}X_t+\tilde{\tau}_{nt}w_tH_t-\tilde{G}_t$.

There are two types of shocks that affect the production and investment efficiency process, contemporary and lagged shocks. In particular:

$$\begin{aligned}
\ln\tilde{z}_t &= \left(1-\rho_z\right)\ln\tilde{z}_{ss}+\rho_z\ln\tilde{z}_{t-1}+\epsilon_t+\nu_{t-1} \\
\ln z_t^x &= \left(1-\rho_x\right)\ln z_{ss}^x+\rho_x\ln z_{t-1}^x+\epsilon_t^x+\nu_{t-1}^x
\end{aligned}$$

11.1 FOC

The FOC of an individual household are:

$$\begin{aligned}
\frac{\psi}{1-h_t} &= \frac{1}{c_t}\left(1-\tilde{\tau}_{nt}\right)w_t \\
\left(1+\tau_{xt}\right)\left(1+\gamma_n\right)\left(1+\gamma_z\right)\frac{\lambda_t}{z_t^x} &= \beta\left(1+\gamma_n\right)\lambda_{t+1}\left(r_{t+1}+\frac{\left(1+\tilde{\tau}_{xt+1}\right)\left(1-\delta\right)}{z_{t+1}^x}\right)
\end{aligned}$$

From the FOC of the firm one gets:

$$r_t = \alpha \left(\frac{\tilde{z}_t H_t}{K_t} \right)^{1-\alpha} \qquad w_t = (1-\alpha)\,\tilde{z}_t \left(\frac{\tilde{z}_t H_t}{K_t} \right)^{-\alpha}$$

Since all households are identical it follows that, in equilibrium:

$$K_t = k_t \quad H_t = h_t \quad X_t = x_t$$

Then the resource constraint of the economy is:

$$k_t^{\alpha} (\tilde{z}_t h_t)^{1-\alpha} + \frac{1}{z_t^x} (1-\delta)\, k_t = c_t + \tilde{G}_t + \frac{1}{z_t^x} (1+\gamma_n)(1+\gamma_z)\, k_{t+1}$$

The exogenous processes follow:

$$S_t = \tilde{P}_0 + \tilde{P}_1 S_{t-1} + \tilde{Q}\epsilon_t$$

The FOC are then:

$$0 = \frac{(1-\tilde{\tau}_{nt})(1-\alpha)\,\tilde{z}_t \left(\frac{\tilde{z}_t h_t}{k_t}\right)^{-\alpha}}{c_t} - \frac{\psi}{1-h_t} \tag{11.1}$$

$$0 = \frac{\beta}{c_{t+1}} \left(\alpha \left(\frac{\tilde{z}_t h_t}{k_t} \right)^{1-\alpha} + \frac{(1+\tilde{\tau}_{xt+1})(1-\delta)}{z_{t+1}^x} \right) - \frac{(1+\tilde{\tau}_{xt})(1+\gamma_z)}{z_t^x c_t} \tag{11.2}$$

$$0 = k_t^{\alpha} (\tilde{z}_t h_t)^{1-\alpha} + \frac{(1-\delta)\, k_t - (1+\gamma_n)(1+\gamma_z)\, k_{t+1}}{z_t^x} - c_t - \tilde{G}_t \tag{11.3}$$

11.2 (Non-Stochastic) Steady state

The (non-stochastic) steady state of the model can be obtained from the conditions above. First the exogenous processes satisfy:

$$\tilde{S}_{ss} = \left(I - \tilde{P} \right)^{-1} \tilde{P}_0$$

From the FOC of the household:

$$\begin{aligned}
\frac{(1+\tilde{\tau}_{xss})(1+\gamma_z)}{z_{ss}^x} &= \beta \left(r_{ss} + \frac{(1+\tilde{\tau}_{xss})(1-\delta)}{z_{ss}^x} \right) \\
\frac{(1+\tilde{\tau}_{xss})}{z_{ss}^x} \left(\frac{(1+\gamma_z)}{\beta} - (1-\delta) \right) &= \alpha \left(\frac{\tilde{z}_{ss} h_{ss}}{k_{ss}} \right)^{1-\alpha} \\
\left(\left(\frac{1+\tilde{\tau}_{xss}}{\alpha z_{ss}^x} \right) \left(\frac{(1+\gamma_z)}{\beta} - \frac{(1-\delta)}{z_t^x} \right) \right)^{\frac{1}{1-\alpha}} &= \frac{\tilde{z}_{ss} h_{ss}}{k_{ss}} \\
\Lambda_1 &= \frac{\tilde{z}_{ss} h_{ss}}{k_{ss}}
\end{aligned}$$

This implies:

$$w_{ss} = (1-\alpha)\,\tilde{z}_{ss} \left(\frac{\tilde{z}_{ss} h_{ss}}{k_{ss}} \right)^{-\alpha} = (1-\alpha)\,\tilde{z}_{ss} \Lambda_1^{-\alpha}$$

One can set the value of $\tilde{G}_{ss}$ so that is some given percentage of the output in steady state, that way: $\tilde{G}_{ss} = \phi_g Y_{ss} = \phi_g k_{ss}^{\alpha} (\tilde{z}_{ss} h_{ss})^{1-\alpha}$.

From the resource constraint consumption satisfies:

$$\begin{aligned} c_{ss} &= k_{ss}^{\alpha}\left(\tilde{z}_{ss}h_{ss}\right)^{1-\alpha}+\left(1-\delta-\left(1+\gamma_{n}\right)\left(1+\gamma_{z}\right)\right)\frac{k_{ss}}{z_{ss}^{x}}-\tilde{G}_{ss} \\ &= \left(1-\phi_{g}\right)k_{ss}^{\alpha}\left(\tilde{z}_{ss}h_{ss}\right)^{1-\alpha}+\left(1-\delta-\left(1+\gamma_{n}\right)\left(1+\gamma_{z}\right)\right)\frac{k_{ss}}{z_{ss}^{x}} \\ &= \left(\left(1-\phi_{g}\right)\Lambda_{1}^{1-\alpha}+\left(\frac{1-\delta-\left(1+\gamma_{n}\right)\left(1+\gamma_{z}\right)}{z_{ss}^{x}}\right)\right)k_{ss} \\ &= \Lambda_{2}k_{ss} \end{aligned}$$

Replacing one gets:

$$\begin{aligned} \frac{\psi c_{ss}}{1-h_{ss}} &= \left(1-\tilde{\tau}_{nss}\right)w_{ss} \\ \psi\Lambda_{2}k_{ss} &= \left(1-\tilde{\tau}_{nss}\right)w_{ss}\left(1-h_{ss}\right) \\ \psi\Lambda_{2}k_{ss}+\left(1-\tilde{\tau}_{nss}\right)w_{ss}h_{ss} &= \left(1-\tilde{\tau}_{nss}\right)w_{ss} \\ \left(\psi\Lambda_{2}+\frac{\left(1-\tilde{\tau}_{nss}\right)w_{ss}\Lambda_{1}}{\tilde{z}_{ss}}\right)k_{ss} &= \left(1-\tilde{\tau}_{nss}\right)w_{ss} \end{aligned}$$

$$k_{ss}=\left[\psi\Lambda_{2}+\frac{\left(1-\tilde{\tau}_{nss}\right)w_{ss}\Lambda_{1}}{\tilde{z}_{ss}}\right]^{-1}\left(1-\tilde{\tau}_{nss}\right)w_{ss}$$

12 Equivalence Result

In order to get the equivalence result consider first three special cases of the prototype economy of last section. In all of them the labor, investment and government spending wedge will be turned off, so that $\tilde{\tau}_{nt} = \tilde{\tau}_{nss}$, $\tilde{\tau}_{xt} = \tilde{\tau}_{xss}$, $\tilde{G}_t = \tilde{G}_{ss}$. The cases to be presented are:

1. There are only efficiency shocks but they are perfectly forecastable one period in advance, that is:
$$\ln \tilde{z}_t = (1-\rho_z)\ln \tilde{z}_{ss} + \rho_z \ln \tilde{z}_{t-1} + \nu_{t-1} \qquad z_t^x = z_{ss}^x$$
2. There are only investment efficiency shocks that with no lagged shocks:
$$\ln z_t^x = (1-\rho_x)\ln z_{ss}^x + \rho_x \ln z_{t-1}^x + \epsilon_t^x \qquad \tilde{z}_t = \tilde{z}_{ss}$$
3. There are product and investment efficiency shocks with current and lagged shocks:
$$\begin{aligned}\ln \tilde{z}_t &= (1-\rho_z)\ln \tilde{z}_{ss} + \rho_z \ln \tilde{z}_{t-1} + \epsilon_t + \nu_{t-1}\\ \ln z_t^x &= (1-\rho_x)\ln z_{ss}^x + \rho_x \ln z_{t-1}^x + \epsilon_t^x + \nu_{t-1}^x\end{aligned}$$

12.1 News shocks in efficiency

Let $z_t = \tilde{z}_t$ and $\tau_{nt} = \tilde{\tau}_{nt}$. From equation (6.1) of the prototype economy one gets:

$$\frac{(1-\tau_{nt})(1-\alpha) z_t \left(\frac{z_t h_t}{k_t}\right)^{-\alpha}}{c_t} = \frac{\psi}{1-h_t} \quad \longrightarrow \quad \frac{(1-\tilde{\tau}_{nt})(1-\alpha)\tilde{z}_t \left(\frac{\tilde{z}_t h_t}{k_t}\right)^{-\alpha}}{c_t} = \frac{\psi}{1-h_t}$$

Which is equation (11.1) of the news economy.

Let

$$G_t = \tilde{G}_{ss} - \frac{(1-\delta)k_t - (1+\gamma_n)(1+\gamma_z)k_{t+1}}{z_t^x} + ((1-\delta)k_t - (1+\gamma_n)(1+\gamma_z)k_{t+1})$$

Then from equation (6.3) of the prototype economy one gets:

$$\begin{aligned}k_t^\alpha (z_t h_t)^{1-\alpha} + (1-\delta)k_t &= c_t + G_t + (1+\gamma_n)(1+\gamma_z)k_{t+1}\\ k_t^\alpha (\tilde{z}_t h_t)^{1-\alpha} + \frac{(1-\delta)k_t}{z_t^x} &= c_t + \tilde{G}_t + \frac{(1+\gamma_n)(1+\gamma_z)k_{t+1}}{z_t^x}\end{aligned}$$

Which is equation (11.3) of the news economy.

Finally the investment wedge is defined implicitly by:

$$\frac{(1+\tau_{xt})(1+\gamma_z)}{\tilde{c}_t} = \frac{\beta}{\tilde{c}_{t+1}}\left(\alpha\left(\frac{z_{t+1}\tilde{h}_{t+1}}{\tilde{k}_{t+1}}\right)^{1-\alpha} + (1+\tau_{xt+1})(1-\delta)\right) \tag{12.1}$$

Where $\left\{\tilde{c}_t, \tilde{h}_t, \tilde{k}_t\right\}$ are the equilibrium allocations of the news economy and $\{z_t\}$ is the series the efficiency wedge in the prototype economy.

The reason the investment wedge is not given by $\tau_{xt} = \frac{1+\tilde{\tau}_{xt}}{z_t^x} - 1$ as one could have imagined from the household's budget constraint is that the agents in the news economy are taking expectations over future values of $\tilde{z}$ and since the one step forecast error is zero always they take different actions when facing $\tilde{z}$ and z, there is forecast error involved in the one-step ahead forecast of z. τ_{xt} must guarantee the equivalence of both models.

This shows that a news shocks presents itself as an efficiency and investment wedge leaving unaltered the labor wedge. The change in the government wedge only depends on the value of the investment efficiency wedge, note that if $z_{ss}^x = 1$ then $G_t = \tilde{G}_t$.

12.2 Investment efficiency wedge

As above its clear that if $z_t = \tilde{z}_t$ and $\tau_{nt} = \tilde{\tau}_{nt}$ then equations (6.1) and (11.1) are identical. In the same way the relation between G_t and $\tilde{G}_t$ (and z_t^x) remains unchanged so as to guarantee that equations (6.3) and (11.3) are identical. Finally if $\tau_{xt} = \frac{1+\tilde{\tau}_{xt}}{z_t^x} - 1$ one gets from equation (6.2):

$$\frac{(1+\tau_{xt})(1+\gamma_z)}{\tilde{c}_t} = \frac{\beta}{\tilde{c}_{t+1}} \left(\alpha \left(\frac{z_{t+1}\tilde{h}_{t+1}}{\tilde{k}_{t+1}} \right)^{1-\alpha} + (1+\tau_{xt+1})(1-\delta) \right)$$

Which turns into:

$$\frac{(1+\tilde{\tau}_{xt})}{z_t^x} \frac{(1+\gamma_z)}{c_t} = \frac{\beta}{c_{t+1}} \left(\alpha \left(\frac{\tilde{z}_{t+1} h_{t+1}}{k_{t+1}} \right)^{1-\alpha} + \frac{(1+\tilde{\tau}_{xt+1})}{z_{t+1}^x} (1-\delta) \right)$$

Which is equation (11.2). Then both economies are equivalent.

An investment efficiency shock presents itself as both an investment wedge and a government spending wedge.

12.3 News in efficiency and investment efficiency wedges

From above when both wedges move and are subject to news shocks one can define all the wedges in the same way they were defined when there are only news shocks to the efficiency wedge. In this case:

$$z_t = \tilde{z}_t \qquad \tau_{nt} = \tilde{\tau}_{nt}$$

$$G_t = \tilde{G}_{ss} - \frac{(1-\delta) k_t - (1+\gamma_n)(1+\gamma_z) k_{t+1}}{z_t^x} + ((1-\delta) k_t - (1+\gamma_n)(1+\gamma_z) k_{t+1})$$

And τ_{xt} is defined implicitly so that equation (12.1) holds. Note that there is no change in the labor wedge.

13 Wedges

Once one has estimated all the unknown parameters of the model it is possible to recover the wedges from observed data. The estimation is discussed below in Part IV. From national accounts one has access to output (y_t), consumption (c_t), investment (x_t) and labor (h_t) series for periods $t = 1, 2, \ldots, T$..

First one has to recover the capital series using the investment. Let $k_1 = k_{ss}(1+\gamma_n)(1+\gamma_z)$, then one can define the capital series recursively as:

$$k_{t+1} = x_t + (1-\delta) k_t$$

Note that these series are in aggregate terms and have not been detrended to eliminate population or technology growth.

Then the government wedge can be obtained as:

$$g_t = y_t - c_t - x_t$$

and the efficiency wedge as:

$$z_t = \left(\frac{y_t}{k_t^\alpha \left((1+\gamma_z)^t h_t \right)^{1-\alpha}} \right)^{\frac{1}{1-\alpha}}$$

and the labor wedge as:

$$\tau_{nt} = 1 - \frac{\psi c_t}{(1-h_t)(1-\alpha) k_t^\alpha \left((1+\gamma_z)^t z_t \right)^{1-\alpha} h_t^{-\alpha}}$$

The investment wedge is treated differently and is recovered from the log-linear approximation of the model (this allows to compute the expectation in the intertemporal first order condition.

The intertemporal condition is:

$$\begin{aligned}
0 &= E_t \left[\frac{\beta}{c_{t+1}} \left(\alpha \left(\frac{z_{t+1} h_{t+1}}{k_{t+1}} \right)^{1-\alpha} + (1+\tau_{xt+1})(1-\delta) \right) \right] - \frac{(1+\tau_{xt})(1+\gamma_z)}{c_t} \\
0 &= E_t \left[\beta e^{-\ln c_{t+1}} \left(\alpha e^{(1-\alpha)(\ln z_{t+1} - \ln k_{t+1})} h_{t+1}^{1-\alpha} + (1+\tau_{xt+1})(1-\delta) \right) \right] - (1+\tau_{xt})(1+\gamma_z) e^{-\ln c_t} \\
0 &= E_t[A] - B
\end{aligned}$$

For convenience consider the linear expansion of the first term only:

$$\begin{aligned}
A &= \beta e^{-\ln c_{t+1}} \left(\alpha e^{(1-\alpha)(\ln z_{t+1} - \ln k_{t+1})} h_{t+1}^{1-\alpha} + (1+\tau_{xt+1})(1-\delta) \right) \\
A &\approx \frac{\beta}{c} \left[- \left(\alpha \left(\frac{zh}{k} \right)^{1-\alpha} + (1+\tau_x)(1-\delta) \right) \hat{c}_{t+1} + (1-\alpha) \alpha \left(\frac{zh}{k} \right)^{1-\alpha} \left(\hat{z}_{t+1} - \hat{k}_{t+1} + h^{-1} \hat{h}_{t+1} \right) + (1-\delta) \hat{\tau}_{xt+1} \right] \\
A &\approx \frac{\beta}{c} \left[- \frac{(1+\tau_{xss})(1+\gamma_z)}{\beta} \hat{c}_{t+1} + (1-\alpha) \alpha \left(\frac{zh}{k} \right)^{1-\alpha} \left(\hat{z}_{t+1} - \hat{k}_{t+1} + h^{-1} \hat{h}_{t+1} \right) + (1-\delta) \hat{\tau}_{xt+1} \right]
\end{aligned}$$

The second term yields:

$$\begin{aligned}
B &= (1+\tau_{xt})(1+\gamma_z) e^{-\ln c_t} \\
B &\approx \frac{(1+\gamma_z)}{c} \hat{\tau}_{xt} - \frac{(1+\tau_{xss})(1+\gamma_z)}{c} \hat{c}_t
\end{aligned}$$

Joining:

$$0 \approx E_t\left[\frac{\beta}{(1+\gamma_z)}\left((1-\alpha)\alpha\left(\frac{zh}{k}\right)^{1-\alpha}\left(\hat{z}_{t+1}-\hat{k}_{t+1}+h^{-1}\hat{h}_{t+1}\right)+(1-\delta)\hat{\tau}_{xt+1}\right)\right]-\hat{\tau}_{xt}+(1+\tau_{xss})\left(\hat{c}_t-E_t\left[\hat{c}_{t+1}\right]\right) \quad (13.1)$$

Finally one can use the solution of the model to obtain the current investment wedge since:

$$\begin{aligned}
E_t\left[\hat{k}_{t+1}\right] &= \hat{k}_{t+1}\\
E_t\left[\hat{z}_{t+1}\right] &= P_1\hat{s}_t\\
E_t\left[\hat{\tau}_{xt+1}\right] &= P_3\hat{s}_t\\
E_t\left[\hat{h}_{t+1}\right] &= C_1\hat{k}_{t+1}+D_1P\hat{s}_t\\
E_t\left[\hat{c}_{t+1}\right] &= C_2\hat{k}_{t+1}+D_2P\hat{s}_t
\end{aligned}$$

Finally note that:g

$$P_i\hat{s}_t = P_{i1}\hat{z}_t+P_{i2}\hat{\tau}_{nt}+P_{i3}\hat{\tau}_{xt}+P_{i4}\hat{g}_t$$

And then:

$$D_jP\hat{s}_t=\sum_{i=1}^{4}D_{ji}\left(P_{i1}\hat{z}_t+P_{i2}\hat{\tau}_{nt}+P_{i3}\hat{\tau}_{xt}+P_{i4}\hat{g}_t\right)$$

Replacing the previous results on (13.1) one can then solve for $\hat{\tau}_{xt}$:

$$\begin{aligned}
\hat{\tau}_{xt} &\approx \frac{\beta}{(1+\gamma_z)}\left((1-\alpha)\alpha\left(\frac{zh}{k}\right)^{1-\alpha}\left(P_1\hat{s}_t-\hat{k}_{t+1}+h^{-1}\left(C_1\hat{k}_{t+1}+D_1P\hat{s}_t\right)\right)+(1-\delta)P_3\hat{s}_t\right)\\
&\quad+(1+\tau_{xss})\left(\hat{c}_t-C_2\hat{k}_{t+1}-D_2P\hat{s}_t\right)\\
\hat{\tau}_{xt} &\approx \frac{\beta}{(1+\gamma_z)}\left(\left((1-\alpha)\alpha\left(\frac{zh}{k}\right)^{1-\alpha}\left(P_1+h^{-1}D_1P\right)+(1-\delta)P_3\right)\hat{s}_t+(1-\alpha)\alpha\left(\frac{zh}{k}\right)^{1-\alpha}\left(h^{-1}C_1-1\right)\hat{k}_{t+1}\right)\\
&\quad+(1+\tau_{xss})\left(\hat{c}_t-C_2\hat{k}_{t+1}-D_2P\hat{s}_t\right)\\
\hat{\tau}_{xt} &\approx \left(\frac{\beta}{(1+\gamma_z)}\left((1-\alpha)\alpha\left(\frac{zh}{k}\right)^{1-\alpha}\left(P_1+h^{-1}D_1P\right)+(1-\delta)P_3\right)-(1+\tau_{xss})D_2P\right)\hat{s}_t\\
&\quad+\left(\frac{\beta(1-\alpha)\alpha}{(1+\gamma_z)}\left(\frac{zh}{k}\right)^{1-\alpha}\left(h^{-1}C_1-1\right)-(1+\tau_{xss})C_2\right)\hat{k}_{t+1}+(1+\tau_{xss})\hat{c}_t\\
\hat{\tau}_{xt} &\approx \Gamma\hat{s}_t+\left(\frac{\beta(1-\alpha)\alpha}{(1+\gamma_z)}\left(\frac{zh}{k}\right)^{1-\alpha}\left(h^{-1}C_1-1\right)-(1+\tau_{xss})C_2\right)\hat{k}_{t+1}+(1+\tau_{xss})\hat{c}_t\\
\hat{\tau}_{xt} &\approx \frac{1}{1-\Gamma_3}\left[\Gamma_1\hat{z}_t+\Gamma_2\hat{\tau}_{nt}+\Gamma_4\hat{g}_t+\left(\frac{\beta(1-\alpha)\alpha}{(1+\gamma_z)}\left(\frac{zh}{k}\right)^{1-\alpha}\left(h^{-1}C_1-1\right)-(1+\tau_{xss})C_2\right)\hat{k}_{t+1}+(1+\tau_{xss})\hat{c}_t\right]
\end{aligned}$$

Where:

$$\Gamma=\left(\frac{\beta}{(1+\gamma_z)}\left((1-\alpha)\alpha\left(\frac{zh}{k}\right)^{1-\alpha}\left(P_1+h^{-1}D_1P\right)+(1-\delta)P_3\right)-(1+\tau_{xss})D_2P\right)$$

Note that in order to apply this formula is necessary to detrend, log and express in deviations form steady state all the series.

14 BCA

Once one has series for all the wedges one can form a vector of states $\hat{s}_t = [\hat{z}_t, \hat{\tau}_{nt}, \hat{\tau}_{xt}, \hat{g}_t]'$. The BCA procedure aims to determine the direct effect of each wedge, while separating it from the forecast effect. To do this consider the following variant of the prototype model in which the variables s_t and $[z_t, \tau_{nt}, \tau_{xt}, g_t]$ are separated. Variables in s_t will be regarded as exogenous states for the model and their evolution will be given by: $\hat{s}_{t+1} = P\hat{s}_t + Q\epsilon_{t+1}$. The wedges $[z_t, \tau_{nt}, \tau_{xt}, g_t]$ will be treated as part of the controls vector d and will be determined by:

$$\begin{bmatrix} \hat{z}_t \\ \hat{\tau}_{nt} \\ \hat{\tau}_{xt} \\ \hat{g}_t \end{bmatrix} = \Lambda \hat{s}_t = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & & \vdots \\ \vdots & & \lambda_3 & 0 \\ 0 & \cdots & 0 & \lambda_4 \end{bmatrix} \hat{s}_t$$

Where Λ is a diagonal that determines which of the states in s have a direct effect over the model and which are only used for forecasting other states. In particular the prototype model has $\Lambda = I_4$ since all wedges are active. If the BCA procedure is performed to establish the direct effect of the efficiency wedge alone then $\lambda_1 = 1$ and $\lambda_2 = \lambda_3 = \lambda_4 = 0$. This implies that although all states in s take their recovered values only $\hat{z}_t$ is affected by them.

The set of equation to linearize is then:

$$0 = \frac{(1-\tau_{nt})(1-\alpha) z_t \left(\frac{z_t h_t}{k_t}\right)^{-\alpha}}{c_t} - \frac{\psi}{1-h_t} \tag{14.1}$$

$$0 = \frac{\beta}{c_{t+1}} \left(\alpha \left(\frac{z_{t+1} h_{t+1}}{k_{t+1}} \right)^{1-\alpha} + (1+\tau_{xt+1})(1-\delta) \right) - \frac{(1+\tau_{xt})(1+\gamma_z)}{c_t} \tag{14.2}$$

$$0 = k_t^\alpha (z_t h_t)^{1-\alpha} + (1-\delta) k_t - c_t - G_t - (1+\gamma_n)(1+\gamma_z) k_{t+1} \tag{14.3}$$

$$0 = z_t - \lambda_1 s_t^1 \tag{14.4}$$

$$0 = \tau_{nt} - \lambda_2 s_t^2 \tag{14.5}$$

$$0 = \tau_{xt} - \lambda_3 s_t^3 \tag{14.6}$$

$$0 = g_t - \lambda_4 s_t^4 \tag{14.7}$$

Where the only exogenous states are $\left[s_t^1, \ldots, s_t^4\right]$.

The decomposition is then obtain in the following way:

1. Obtain the full set of parameters of the prototype economy. (This involves maximum likelihood).
2. Recover the capital series.
   (a) Set an initial value $k_1 = k_{ss}(1+\gamma_n)(1+\gamma_z)$
   (b) Iterate forward: $k_{t+1} = x_t + (1-\delta) k_t$
3. Recover series for all wedges:
   (a) $g_t = y_t - c_t - x_t$
   (b) $z_t = \left(\frac{y_t}{k_t^\alpha \left((1+\gamma_z)^t h_t\right)^{1-\alpha}} \right)^{\frac{1}{1-\alpha}}$
   (c) $\tau_{nt} = 1 - \frac{\psi c_t}{(1-h_t)(1-\alpha) k_t^\alpha \left((1+\gamma_z)^t z_t\right)^{1-\alpha} h_t^{-\alpha}}$

(d) $\hat{\tau}_{xt} \approx \frac{1}{1-\Gamma_3}\left[\Gamma_1 \hat{z}_t + \Gamma_2 \hat{\tau}_{nt} + \Gamma_4 \hat{g}_t + \left(\frac{\beta(1-\alpha)\alpha}{(1+\gamma_z)}\left(\frac{zh}{k}\right)^{1-\alpha}\left(h^{-1}C_1 - 1\right) - (1+\tau_{xss}) C_2\right)\hat{k}_{t+1} + (1+\tau_{xss})\hat{c}_t\right]$

Where:

$$\Gamma = \frac{\beta}{(1+\gamma_z)}\left((1-\alpha)\alpha\left(\frac{zh}{k}\right)^{1-\alpha}\left(P_1 + h^{-1}D_1 P\right) + (1-\delta) P_3\right) - (1+\tau_{xss}) D_2 P$$

4. Re-label the wedges as:

$$\begin{bmatrix} \hat{s}_t^1 \\ \hat{s}_t^2 \\ \hat{s}_t^3 \\ \hat{s}_t^4 \end{bmatrix} = \begin{bmatrix} \hat{z}_t \\ \hat{\tau}_{nt} \\ \hat{\tau}_{xt} \\ \hat{g}_t \end{bmatrix}$$

5. For the effect of the i^{th} wedge do:

   (a) Set $\lambda_i = 1$ and $\lambda_j = 0$ for $j \neq 0$.

   (b) Solve the linear approximation to the set of equations (14.1) to (14.7) to obtain a solution of the form:

$$\begin{aligned} \hat{k}_{t+1}^i &= A^i \hat{k}_t + B^i \hat{s}_t \\ \hat{d}_t^i &= C^i \hat{k}_t + D^i \hat{s}_t \end{aligned}$$

   Where $d = [h, c, z, \tau_n, \tau_x, g]^{'}$.

   (c) Let $\hat{k}_1 = 0$ and $\hat{s}_t$ be given by the series defined in part (4).

   (d) Use the policy function to simulate variables.

Part IV
Estimation of the model

The objective is to estimate by maximum likelihood the parameters of a model whose solution can be represented as a VAR process. Since not all of the variables in the VAR are observable (for example the wedges) the parameters that form the VAR matrices cannot be estimated directly. The Kalman filter allows to overcome this by embedding the VAR in a state space system.

A state space system is conformed by a transition equation and measurement equation. Two types of variables are considered in the system, "states" and "observables", this "states" are not the same thing that was called a state when solving the model, here the "states" are -potentially- unobserved variables that affect the observable variables. The measurement equation describes the relation between states and observables, while the transition equation describes the dynamics of the states.

As will be shown below one way to proceed is to use the VAR representation of the model's solution to form the transition equation and then relate those variables to observables using the measurement equation.

The Kalman filter allows to recover series for the unobserved variables and also generates a likelihood function for the model, one can then use this likelihood function to estimate parameters of interest.

The following two sections cover the construction of the Kalman filter from the solution of the model (the linear policy functions) and then how to use the Kalman filter to obtain the model's likelihood function.

15 Kalman Filter

15.1 Model's solution

Let k_t be a vector of the endogenous states of the model, in the prototype economy of homework 2 the only endogenous state is capital but k_t is in general a $n_k \times 1$ vector[1]. Let s_t be a vector of the exogenous states of the model of size $n_s \times 1$. Let d_t be a $n_d \times 1$ vector containing the model's decision (or control) variables and the prices, for example labor, consumption, wages, etc.

The solution of the model is formed by three equations:

$$\begin{aligned} \hat{k}_{t+1} &= A_m \hat{k}_t + B_m \hat{s}_t \\ \hat{d}_t &= C_m \hat{k}_t + D_m \hat{s}_t \\ \hat{s}_{t+1} &= P \hat{s}_t + Q \epsilon_{t+1} \end{aligned}$$

Where $\hat{x} = x - x_{ss}$ and, for simplicity, it is assumed that $\epsilon \sim (0, I)$ is of dimension $n_s \times 1$, this is not necessary since matrix Q can be adapted for other sizes of ϵ. The above equations can be obtained by solving the linear quadratic approximation to the original problem or by solving the first order approximation of the original FOC of the problem.

Note that usually d contains only one or two decision variables (h and c for example), and that the other decision variables (like investment and prices) have been replaced out of the system using their definitions. This definitions can be (log) linearized to obtain a system like:

$$\hat{d}_t^2 = \alpha_1 \hat{k}_t + \alpha_2 \hat{k}_{t+1} + \alpha_3 \hat{d}_t + \alpha_4 E\left[\hat{d}_{t+1}\right] + \alpha_5 \hat{s}_t + \alpha_6 E\left[\hat{s}_{t+1}\right]$$

Where d^2 is a vector listing all other decisions and prices that one wishes to include in the d vector, and α_i are matrices of the right dimension. Once this is done one can replace for the solutions above

[1] Depending on the solution method $k_t = \ln k_t$ or $k_t = k_t$, the same thing goes for all the other variables.

to get:

$$
\begin{aligned}
\hat{d}_t^2 &= (\alpha_1 + \alpha_2 A_m + \alpha_3 C_m)\hat{k}_t + (\alpha_5 + \alpha_2 B_m \hat{s}_t + \alpha_3 D_m)\hat{s}_t + \alpha_4 E\left[\hat{d}_{t+1}\right] + \alpha_6 E\left[\hat{s}_{t+1}\right] \\
&= (\alpha_1 + \alpha_2 A_m + \alpha_3 C_m)\hat{k}_t + (\alpha_5 + \alpha_2 B_m \hat{s}_t + \alpha_3 D_m + \alpha_6 P)\hat{s}_t + \alpha_4 E\left[C_m \hat{k}_{t+1} + D_m \hat{s}_{t+1}\right] \\
&= (\alpha_1 + \alpha_2 A_m + \alpha_3 C_m + \alpha_4 C_m A_m)\hat{k}_t + (\alpha_5 + \alpha_2 B_m \hat{s}_t + \alpha_3 D_m + \alpha_6 P + \alpha_4 (C_m B_m + D_m P))\hat{s}_t
\end{aligned}
$$

This gives a dynamic for d^2 as a function of k and s and then can be stacked below d in the equations above.

When solving the linear approximation to the FOC of the problem one can omit the above calculations by including the (non-linear) definitions of the variables in d^2 as part of the FOC system. The solution of the model will then include the dynamics of all the variables in d^2 as part of d with no need of any extra computations.

The solution to the model can be then express as a VAR by first noting that:

$$
\begin{aligned}
\hat{d}_{t+1} &= C_m \hat{k}_{t+1} + D_m \hat{s}_{t+1} \\
&= C_m \left(A_m \hat{k}_t + B_m \hat{s}_t\right) + D_m \left(P \hat{s}_t + Q \epsilon_{t+1}\right) \\
&= C_m A_m \hat{k}_t + (C_m B_m + D_m P)\hat{s}_t + D_m Q \epsilon_{t+1}
\end{aligned}
$$

Which gives:

$$
\begin{bmatrix} \hat{k}_{t+1} \\ \hat{d}_{t+1} \\ \hat{s}_{t+1} \end{bmatrix} = \begin{bmatrix} A_m & 0_{n_x \times n_d} & B_m \\ C_m A_m & 0_{n_d \times n_d} & C_m B_m + D_m P \\ 0_{n_s \times n_x} & 0_{n_s \times n_d} & P \end{bmatrix} \begin{bmatrix} \hat{k}_{t+1} \\ \hat{d}_{t+1} \\ \hat{s}_{t+1} \end{bmatrix} + \begin{bmatrix} 0_{n_x \times n_s} \\ D_m Q \\ Q \end{bmatrix} \epsilon_{t+1}
$$

For ease of notation call $x_t = \left[\hat{k}_t', \hat{d}_t', \hat{s}_t'\right]'$ and then define:

$$
x_{t+1} = A x_t + B \epsilon_{t+1}
$$

Where:

$$
A = \begin{bmatrix} A_m & 0_{n_x \times n_d} & B_m \\ C_m A_m & 0_{n_d \times n_d} & C_m B_m + D_m P \\ 0_{n_s \times n_x} & 0_{n_s \times n_d} & P \end{bmatrix} \qquad B = \begin{bmatrix} 0_{n_x \times n_s} \\ D_m Q \\ Q \end{bmatrix}
$$

15.2 State Space

As mentioned above all the model variables (endogenous states, exogenous states and decisions) are treated as "states" of the state space system. The VAR obtained above forms the transition equation of the state space since it characterizes the dynamics of the states of the model. The other equation that completes the state space is the measurement equation:

$$
y_t = C x_t + \omega_t
$$

Where y_t is a vector of size $n_y \times 1$ of observed variables and ω_t is a vector of (possibly) serially correlated measurement errors that follow:

$$
\omega_{t+1} = D\omega_t + \eta_{t+1} \qquad \eta_t \sim iid\,(0, R)
$$

Matrix C is determined by which variables are included in the observable set, usually it only has zeros and ones that link an observed variable to its equivalent in vector x (for example consumption or GDP).

For simplicity in what follows define an auxiliary variable $\overline{y}_t = y_{t+1} - Dy_t$ so that:

$$\begin{aligned}
\overline{y}_t &= y_{t+1} - Dy_t \\
&= Cx_{t+1} + \omega_{t+1} - D\left(Cx_t + \omega_t\right) \\
&= C\left(Ax_t + B\epsilon_{t+1}\right) - DCx_t + \left(\omega_{t+1} - D\omega_t\right) \\
&= \left(CA - DC\right)x_t + CB\epsilon_{t+1} + \eta_{t+1} \\
\overline{y}_t &= \overline{C}x_t + CB\epsilon_{t+1} + \eta_{t+1}
\end{aligned}$$

Where $\overline{C} = CA - DC$. The state space system is then:

$$\begin{aligned}
x_{t+1} &= Ax_t + B\epsilon_{t+1} \\
\overline{y}_t &= \overline{C}x_t + CB\epsilon_{t+1} + \eta_{t+1} \\
\omega_{t+1} &= D\omega_t + \eta_{t+1}
\end{aligned}$$

Using this system one can derive formulas for the predictions of x_t given the observed variables y_t (hence $\overline{y}_t$) and an initial guess for the value and variance of x_t. These predictions are obtained through the Kalman filter. Note that all of this is done given a set of parameter values, and the subsequent model solution for those values.

15.3 Kalman Filter

To obtain the predictions first define the projection of Y onto X as:

$$\begin{aligned}
\hat{E}\left[Y|X\right] &= E\left[Y\right] + \Sigma_{xy}\Sigma_{xx}^{-1}\left(X - E\left[X\right]\right) \\
\Sigma_{xx} &= E\left[\left(X - E\left[X\right]\right)\left(X - E\left[X\right]\right)^{'}\right] \\
\Sigma_{xy} &= E\left[\left(Y - E\left[Y\right]\right)\left(X - E\left[X\right]\right)^{'}\right]
\end{aligned}$$

This formula is valid for any variables Y and X. The idea is to apply it to predictions of $\overline{y}_t$ given its past values $\overline{y}^{t-1}$ and a guess for the initial value x. The reason that only an initial value is needed is that further predictions of x are obtained given the observations of $\overline{y}$, so given the information in $\overline{y}^{t-1}$ there is no extra information in the predictions of x.

Define the innovation (or prediction error) as:

$$\begin{aligned}
u_t &= \overline{y}_t - \hat{E}\left[\overline{y}_t|\overline{y}^{t-1}, x_0\right] \\
&= \overline{y}_t - \hat{E}\left[\overline{C}x_t + CB\epsilon_{t+1} + \eta_{t+1}|\overline{y}^{t-1}, x_0\right] \\
&= \overline{y}_t - \overline{C}\hat{E}\left[x_t|\overline{y}^{t-1}, x_0\right] - \hat{E}\left[CB\epsilon_{t+1} + \eta_{t+1}|\overline{y}^{t-1}, x_0\right] \\
&= \overline{y}_t - \overline{C}\hat{E}\left[x_t|\overline{y}^{t-1}, x_0\right] \\
&= \overline{y}_t - \overline{C}\hat{x}_t^{t-1}
\end{aligned}$$

$$\overline{y}_t = \overline{C}\hat{x}_t^{t-1} + u_t$$

Where $\hat{x}_t^{t-1}$ is the prediction for x at time t given information up to time $t-1$. This gives an equation that relates the observed variable $\overline{y}_t$ to the predictions of x (which are observable) and the innovation term. Given a distribution for the innovation term this equation can be used to construct the likelihood function of the model.

For the following results note that:

$$\hat{E}\left[Y|X_1, \ldots X_n\right] = E\left[Y\right] + \sum_{i=1}^{n}\left(\hat{E}\left[Y|X_i\right] - E\left[Y\right]\right) \qquad X_i \perp X_j$$

Also note that since $\left\{\overline{y}^{t-1}, x_0\right\}$ spans the same linear space than $\left\{u^{t-1}, x_0\right\}$ and that the elements in the latter set are orthogonal to each other by construction (since innovations are uncorrelated with each other. Then:

$$\begin{aligned}
\hat{x}_{t+1}^{t} &= \hat{E}\left[x_{t+1}|\overline{y}^{t}, x_0\right] \\
&= \hat{E}\left[x_{t+1}|u^{t}, x_0\right] \\
&= \hat{E}\left[x_{t+1}|u_t, x_0\right] + \left(\hat{E}\left[x_{t+1}|u^{t-1}, x_0\right] - E\left[x_{t+1}\right]\right) \\
&= \hat{E}\left[x_{t+1}|u_t, x_0\right] + \hat{E}\left[Ax_t + B\epsilon_{t+1}|u^{t-1}, x_0\right] - E\left[x_{t+1}\right] \\
&= \hat{E}\left[x_{t+1}|u_t, x_0\right] + A\hat{E}\left[x_t|u^{t-1}, x_0\right] - E\left[x_{t+1}\right] \\
&= \left(E\left[x_{t+1}\right] + \Sigma_{x_{t+1}u_t}\Sigma_{u_tu_t}^{-1}\left(u_t - E\left[u_t\right]\right)\right) + A\hat{x}_t^{t-1} - E\left[x_{t+1}\right] \\
&= A\hat{x}_t^{t-1} + \Sigma_{x_{t+1}u_t}\Sigma_{u_tu_t}^{-1}u_t
\end{aligned}$$

This equation gives a law of motion for the prediction $\hat{x}_t^{t-1}$, to complete the system one needs the covariance matrices $\Sigma_{x_{t+1}u_t}$ and $\Sigma_{u_tu_t}$. For future reference $\Sigma_{u_tu_t} = \Omega_t$.

The expression for the matrices are:

$$\begin{aligned}
\Sigma_{x_{t+1}u_t} &= \text{cov}\left[x_{t+1}, u_t\right] \\
&= \text{cov}\left[Ax_t + B\epsilon_{t+1}\,,\, \overline{y}_t - \overline{C}\hat{x}_t^{t-1}\right] \\
&= \text{cov}\left[Ax_t + B\epsilon_{t+1}\,,\, \overline{C}\left(x_t - \hat{x}_t^{t-1}\right) + CB\epsilon_{t+1} + \eta_{t+1}\right] \\
&= E\left[\left(Ax_t + B\epsilon_{t+1} - E\left[x_{t+1}\right]\right)\left(\overline{C}\left(x_t - \hat{x}_t^{t-1}\right) + CB\epsilon_{t+1} + \eta_{t+1} - E\left[u_t\right]\right)^{'}\right] \\
&= E\left[\left(A\left(x_t - E\left[x_t\right]\right) + B\epsilon_{t+1}\right)\left(\overline{C}\left(x_t - \hat{x}_t^{t-1}\right) + \left(CB\epsilon_{t+1} + \eta_{t+1}\right)\right)^{'}\right] \\
&= E\left[A\left(x_t - E\left[x_t\right]\right)\left(x_t - \hat{x}_t^{t-1}\right)^{'}\overline{C}^{'}\right] + E\left[B\epsilon_{t+1}\left(x_t - \hat{x}_t^{t-1}\right)^{'}\overline{C}^{'}\right] \\
&\quad + E\left[A\left(x_t - E\left[x_t\right]\right)\left(CB\epsilon_{t+1} + \eta_{t+1}\right)^{'}\right] + E\left[B\epsilon_{t+1}\left(CB\epsilon_{t+1} + \eta_{t+1}\right)^{'}\right] \\
&= E\left[A\left(x_t - E\left[x_t\right]\right)\left(x_t - \hat{x}_t^{t-1}\right)^{'}\overline{C}^{'}\right] + BE\left[\epsilon_{t+1}\epsilon_{t+1}\right]B^{'}C^{'} \\
&= A\left(E\left[\left(x_t - E\left[x_t\right]\right)\left(x_t - \hat{x}_t^{t-1}\right)^{'}\right]\right)\overline{C}^{'} + BB^{'}C^{'}
\end{aligned}$$

Where, using the fact that $\hat{x}_t^{t-1}$ is unbiased so that $E\left[x_t\right] = E\left[\hat{x}_t^{t-1}\right]$ and that it is also orthogonal to the prediction error so that $E\left[\hat{x}_t^{t-1}\left(x_t - \hat{x}_t^{t-1}\right)^{'}\right] = 0$:

$$\begin{aligned}
E\left[\left(x_t - E\left[x_t\right]\right)\left(x_t - \hat{x}_t^{t-1}\right)^{'}\right] &= E\left[x_t\left(x_t - \hat{x}_t^{t-1}\right)^{'}\right] - E\left[E\left[x_t\right]\left(x_t - \hat{x}_t^{t-1}\right)^{'}\right] \\
&= E\left[\left(x_t - \hat{x}_t^{t-1} + \hat{x}_t^{t-1}\right)\left(x_t - \hat{x}_t^{t-1}\right)^{'}\right] - E\left[x_t\right]E\left[\left(x_t - \hat{x}_t^{t-1}\right)^{'}\right] \\
&= E\left[\left(x_t - \hat{x}_t^{t-1}\right)\left(x_t - \hat{x}_t^{t-1}\right)^{'}\right] + E\left[\hat{x}_t^{t-1}\left(x_t - \hat{x}_t^{t-1}\right)^{'}\right] \\
&= E\left[\left(x_t - \hat{x}_t^{t-1}\right)\left(x_t - \hat{x}_t^{t-1}\right)^{'}\right] \\
&= \Sigma_t
\end{aligned}$$

Where Σ_t is the variance of the one step prediction error for x_t. Then:

$$\Sigma_{x_{t+1}u_t} = A\Sigma_t\overline{C}^{'} + BB^{'}C^{'}$$

For the covariance matrix of the forecast error u_t:

$$\begin{aligned}\Omega_t &= E\left[\left(\overline{C}\left(x_t-\hat{x}_t^{t-1}\right)+CB\epsilon_{t+1}+\eta_{t+1}\right)\left(\overline{C}\left(x_t-\hat{x}_t^{t-1}\right)+CB\epsilon_{t+1}+\eta_{t+1}\right)^{'}\right]\\ &= \overline{C}\Sigma_t\overline{C}^{'}+CBB^{'}C^{'}+R\end{aligned}$$

Letting

$$K_t=\Sigma_{x_{t+1}u_t}\Omega_t^{-1}=\left(A\Sigma_t\overline{C}^{'}+BB^{'}C^{'}\right)\Omega_t^{-1}$$

This is referred to as the Kalman gain, it determines how much of the forecast error of the observable variables is used to update the prediction for x, note that if this error is too volatile (big Ω) then the forecast error is not used as much, in the same way the more correlation is between x and u the information in u becomes more relevant to update the prediction of x.

One gets the law of motion for the prediction of x given known parameters and the variance of the prediction error for x (Σ_t).

$$\hat{x}_{t+1}^t=A\hat{x}_t^{t-1}+K_tu_t$$

To complete the system one needs to determine Σ_t. Note first:

$$\begin{aligned}x_{t+1}-\hat{x}_{t+1}^t &= \left(Ax_t+B\epsilon_{t+1}\right)-\left(A\hat{x}_t^{t-1}+K_tu_t\right)\\ &= A\left(x_t-\hat{x}_t^{t-1}\right)+B\epsilon_{t+1}-K_t\left(\overline{y}_t-\overline{C}\hat{x}_t^{t-1}\right)\\ &= A\left(x_t-\hat{x}_t^{t-1}\right)+B\epsilon_{t+1}-K_t\left(\overline{C}x_t+CB\epsilon_{t+1}+\eta_{t+1}-\overline{C}\hat{x}_t^{t-1}\right)\\ &= \left(A-K_t\overline{C}\right)\left(x_t-\hat{x}_t^{t-1}\right)+\left(B-K_tCB\right)\epsilon_{t+1}-K_t\eta_{t+1}\end{aligned}$$

Then:

$$\begin{aligned}\Sigma_{t+1} &= E\left[\left(x_{t+1}-\hat{x}_{t+1}^t\right)\left(x_{t+1}-\hat{x}_{t+1}^t\right)^{'}\right]\\ &= \left(A-K_t\overline{C}\right)\Sigma_t\left(A-K_t\overline{C}\right)^{'}+\left(B-K_tCB\right)\left(B-K_tCB\right)^{'}+K_tRK_t^{'}\end{aligned}$$

To further simply it note:

$$\begin{aligned}\left(B-K_tCB\right)\left(B-K_tCB\right)^{'} &= \left(B-K_tCB\right)\left(B^{'}-B^{'}C^{'}K_t^{'}\right)K_tRK_t^{'}\\ &= BB^{'}-K_tCBB^{'}-BB^{'}C^{'}K_t^{'}+K_tCBB^{'}C^{'}K_t^{'}\end{aligned}$$

And:

$$\left(A-K_t\overline{C}\right)\Sigma_t\left(A-K_t\overline{C}\right)^{'}=A\Sigma_tA^{'}-A\Sigma_t\overline{C}^{'}K_t^{'}-K_t\overline{C}\Sigma_tA+K_t\overline{C}\Sigma_t\overline{C}^{'}K_t^{'}$$

Replacing gives:

$$\begin{aligned}\Sigma_{t+1} &= A\Sigma_tA^{'}+BB^{'}-\left(A\Sigma_t\overline{C}^{'}+BB^{'}C^{'}\right)K_t^{'}-K_t\left(\overline{C}\Sigma_tA+CBB^{'}\right)+K_t\left(\overline{C}\Sigma_t\overline{C}^{'}+CBB^{'}C^{'}+R\right)K_t^{'}\\ &= A\Sigma_tA^{'}+BB^{'}-\Sigma_{xu}K_t^{'}-K_t\Sigma_{xu}^{'}+K_t\Omega_tK_t^{'}\\ &= A\Sigma_tA^{'}+BB^{'}-\Sigma_{xu}\Omega_t^{-1}\Sigma_{xu}^{'}-\Sigma_{xu}\Omega_t^{-1}\Sigma_{xu}^{'}+K_t\Omega_tK_t^{'}\\ &= A\Sigma_tA^{'}+BB^{'}-\Sigma_{xu}\Omega_t^{-1}\Sigma_{xu}^{'}-\Sigma_{xu}\Omega_t^{-1}\Sigma_{xu}^{'}+\Sigma_{xu}\Omega_t^{-1}\Omega_t\Omega_t^{-1}\Sigma_{xu}^{'}\\ &= A\Sigma_tA^{'}+BB^{'}-\Sigma_{xu}\Omega_t^{-1}\Sigma_{xu}^{'}\end{aligned}$$

Using the definition of Σ_{xu} and Ω_t one gets:

$$\Sigma_{t+1}=A\Sigma_tA^{'}+BB^{'}-\left(A\Sigma_t\overline{C}^{'}+BB^{'}C^{'}\right)\left(\overline{C}\Sigma_t\overline{C}^{'}+CBB^{'}C^{'}+R\right)^{-1}\left(A\Sigma_t\overline{C}^{'}+BB^{'}C^{'}\right)^{'}$$

This is a recursive formula for Σ. With this the Kalman filter is completely characterized.

15.3.1 Relevant equation Kalman Filter

The relevant equations of the system are:

$$\begin{aligned}
\Omega_t &= \overline{C}\Sigma_t\overline{C}^{'} + CBB^{'}C^{'} + R \\
K_t &= \left(A\Sigma_t\overline{C}^{'} + BB^{'}C^{'}\right)\Omega_t^{-1} \\
u_t &= \overline{y}_t - \overline{C}\hat{x}_t^{t-1} \\
\hat{x}_{t+1}^t &= A\hat{x}_t^{t-1} + K_t u_t \\
\Sigma_{t+1} &= A\Sigma_t A^{'} + BB^{'} - \left(A\Sigma_t\overline{C}^{'} + BB^{'}C^{'}\right)\Omega_t^{-1}\left(A\Sigma_t\overline{C}^{'} + BB^{'}C^{'}\right)^{'}
\end{aligned}$$

The system describes a recursive system that can be used to construct predictions of the unobserved states x by starting with a guess for $\hat{x}_0^{-1}$ and Σ_0. With that guess one can built Ω_t, K_t, u_t and then $\hat{x}_{t+1}^t$ providing a series for predictions of x. It allows to construct Σ_{t+1} that, along with $\hat{x}_{t+1}^t$, enables to repeat the procedure for the next period.

The problem of what to use as initial guesses can be solve by setting $\hat{x}_0^{-1} = 0$, its unconditional mean, and $\Sigma_0 = \overline{\Sigma}$ where $\overline{\Sigma}$ is the solution to the matrix equation:

$$\overline{\Sigma} = A\overline{\Sigma}A^{'} + BB^{'} - \left(A\overline{\Sigma C}^{'} + BB^{'}C^{'}\right)\left(\overline{C\Sigma C}^{'} + CBB^{'}C^{'} + R\right)^{-1}\left(A\overline{\Sigma C}^{'} + BB^{'}C^{'}\right)^{'}$$

Or to the system:

$$\begin{aligned}
\overline{\Omega} &= \overline{C\Sigma C}^{'} + CBB^{'}C^{'} + R \\
\overline{\Sigma} &= A\overline{\Sigma}A^{'} + BB^{'} - \left(A\overline{\Sigma C}^{'} + BB^{'}C^{'}\right)\overline{\Omega}^{-1}\left(A\overline{\Sigma C}^{'} + BB^{'}C^{'}\right)^{'}
\end{aligned}$$

Doing this also eliminates the first, second and last step of the iterative procedure described above since one gets: $\Omega_t = \overline{\Omega}$, $\Sigma_t = \overline{\Sigma}$ and $K_t = \overline{K} = \left(A\overline{\Sigma C}^{'} + BB^{'}C^{'}\right)\overline{\Omega}^{-1}$ for all periods. The matrix equations above can be solve by iteration starting from an arbitrary positive definite matrix.

16 Likelihood function and estimation

16.1 Likelihood function

Under the assumption that $u_t \sim N(0, \Omega_t)$ and noting that $u_i \perp u_j$ for all $i \neq j$ one has that u_i is independent of u_j (since in the normal distribution no-correlation implies independence. From the Kalman filter procedure one obtains a series for $\{u_t\}$ any given u_t has PDF:

$$f(u_t|\Gamma) = (2\pi |\Omega_t|)^{-\frac{1}{2}} e^{-\frac{1}{2}u_t \Omega_t^{-1} u_t^{'}}$$

Where $|\Omega_t|$ is the determinant of matrix Ω_t and Γ is a vector with all the parameters of the model. The likelihood of the sample is (by independence):

$$L(\Gamma|u) = \prod_{t=1}^{T} f(u_t|\Gamma)$$

The log-likelihood function is:

$$l(\Gamma) = \sum_{t=1}^{T} \left(-\frac{1}{2} \ln(2\pi) - \frac{1}{2} \log |\Omega_t| - \frac{1}{2} u_t \Omega_t^{-1} u_t^{'} \right)$$

Note that $u_t \Omega_t^{-1} u_t^{'}$ is a scalar and hence $u_t \Omega_t^{-1} u_t^{'} = \text{tr}\left(u_t \Omega_t^{-1} u_t^{'}\right) = \text{tr}\left(\Omega_t^{-1} u_t^{'} u_t\right)$, where $\text{tr}(\cdot)$ is the trace operator. The negative of the log-likelihood is then proportional to:

$$nl(\Gamma) \propto \sum_{t=1}^{T} \left(\log |\Omega_t| + \text{tr}\left(\Omega_t^{-1} u_t^{'} u_t\right)\right)$$

This function can be minimized to find the maximum likelihood estimator for Γ.

16.2 Estimation Procedure

The estimation procedure is the following:

1. Set a value for parameters Γ_0 (this includes P, Q, C, D, R).
2. Solve the model to get: A_m, B_m, C_m, D_m and with them A and B.
3. Solve for the steady state of the kalman filter $\overline{\Sigma}$, $\overline{\Omega}$ and $\overline{K}$.
4. Use the Kalman filter to get a series for $\{u_t\}$.
5. Evaluate the negative of the likelihood function $nl(\Gamma_0)$
6. Update to Γ_1 (the computer should give you this).
7. Repeat (2) to (6) until convergence in Γ.

Part V

Signal extraction problems

The objective of these Sections is to show the solution of (linear) rational expectations models with imperfect information. In them agents face uncertainty about the realization of exogenous shocks and learn about them through unbiased signals. First a static model of consumption and leisure choice is presented and solved manually. The complete information is covered first as a benchmark, then imperfect information under iid and AR(1) shocks. Then the general solution is considered for two special cases: when all decisions are taken with the same information set and when different decisions use different information sets.

17 A static problem

Consider an agent that has to decide how much to work and consume given a level of productivity.

$$\max \quad \frac{c^{1-\sigma}}{1-\sigma} + \chi \ln(1-n) \qquad \text{s.t. } c = an^{\alpha}$$

Without uncertainty the solution is given by solving the following set of equations:

$$\begin{aligned} -\frac{\chi}{1-n} + c^{-\sigma}\alpha a n^{\alpha-1} &= 0 \\ an^{\alpha} - c &= 0 \end{aligned}$$

One can linearize these FOC around an arbitrary point and solve them:

$$\begin{aligned} -\frac{\chi}{(1-n)^2} n\hat{n} + c^{-\sigma}\alpha a n^{\alpha-1}\left(-\sigma\hat{c} + \hat{a} + (\alpha-1)\hat{n}\right) &\approx 0 \\ \hat{a} + \alpha\hat{n} - \hat{c} &\approx 0 \end{aligned}$$

Which using the non-linear FOC:

$$\begin{aligned} -\frac{n}{1-n}\hat{n} - \sigma\hat{c} + \hat{a} + (\alpha-1)\hat{n} &\approx 0 \\ \hat{a} + \alpha\hat{n} - \hat{c} &\approx 0 \end{aligned}$$

Finally:

$$\begin{aligned} -\frac{n}{1-n}\hat{n} - \sigma\left(\hat{a} + \alpha\hat{n}\right) + \hat{a} + (\alpha-1)\hat{n} &\approx 0 \\ -\frac{n}{1-n}\hat{n} + (1-\sigma)\hat{a} + \left(\alpha(1-\sigma) - 1\right)\hat{n} &\approx 0 \\ (1-n)(1-\sigma)\hat{a} + \left(\alpha(1-n)(1-\sigma) - 1\right)\hat{n} &\approx 0 \end{aligned}$$

The solution is:

$$\begin{aligned} \hat{n} &\approx \frac{(1-n)(1-\sigma)}{1-\alpha(1-n)(1-\sigma)}\hat{a} \\ \hat{c} &= \hat{a} + \alpha\hat{n} \end{aligned}$$

18 A static problem with signals and iid shocks

Now suppose that at the beginning of the period n is chosen without complete knowledge of a. The agent receives an unbiased signal about a. Suppose that $\ln a \sim N\left(0, \kappa^{-1}\right)$ and that the signal is $\ln a^s \sim N\left(\ln a, \kappa_s^{-1}\right)$. After n is chosen the agent observes a and consumes. The problem is now:

$$\max \quad E\left[\frac{c^{1-\sigma}}{1-\sigma} + \chi \ln(1-n) \,|a^s\right] \qquad \text{s.t. } c = an^{\alpha}$$

The FOC are:

$$\begin{aligned} -\frac{\chi}{1-n} + E\left[c^{-\sigma}\alpha a n^{\alpha-1}|a^s\right] &= 0 \\ an^{\alpha} - c &= 0 \end{aligned}$$

As before one can linearize these FOC around an arbitrary point and solve them:

$$\begin{aligned} -\frac{\chi}{(1-n)^2} n\hat{n} + c^{-\sigma}\alpha a n^{\alpha-1} E\left[\left(-\sigma\hat{c} + \hat{a} + (\alpha-1)\hat{n}\right)|a^s\right] &\approx 0 \\ \hat{a} + \alpha\hat{n} - \hat{c} &\approx 0 \end{aligned}$$

And by canceling terms:

$$\begin{aligned} \left((\alpha-1) - \frac{n}{1-n}\right)\hat{n} + E\left[\left(-\sigma\hat{c} + \hat{a}\right)|a^s\right] &\approx 0 \\ \hat{a} + \alpha\hat{n} - \hat{c} &\approx 0 \end{aligned}$$

Guess that the solution has the form:

$$\hat{n} = \gamma\hat{a}^s \qquad \hat{c} = \pi_1\hat{a}^s + \pi_2\hat{a}$$

$$\begin{bmatrix} \hat{n} \\ \hat{c} \end{bmatrix} = \begin{bmatrix} \gamma & 0 \\ \pi_1 & \pi_2 \end{bmatrix} \begin{bmatrix} \hat{a}^s \\ \hat{a} \end{bmatrix}$$

This guess captures the fact that only the signal is available to the agent when choosing labor, but both the signal and the realization of the productivity are available when choosing consumption.

Using the second equation:

$$\left(1-\pi_2\right)\hat{a} + \left(\alpha\gamma - \pi_1\right)\hat{a}^s = 0$$

This equation gives:

$$\pi_1 = 1 \qquad \pi_2 = \alpha\gamma$$

Using the first equation and replacing by consumption:

$$\begin{aligned} \left((\alpha-1) - \frac{n}{1-n}\right)\hat{n} + E\left[\left(-\sigma\hat{c} + \hat{a}\right)|a^s\right] &= 0 \\ \left((\alpha-1) - \frac{n}{1-n}\right)\hat{n} + E\left[\left(-\sigma\left(\hat{a} + \alpha\hat{n}\right) + \hat{a}\right)|a^s\right] &= 0 \\ \left((\alpha-1) - \frac{n}{1-n} - \sigma\alpha\right)\hat{n} + (1-\sigma)\,E\left[\hat{a}|a^s\right] &= 0 \\ \left((1-n)(1-\sigma)\alpha - 1\right)\hat{n} + (1-n)(1-\sigma)\,E\left[\hat{a}|a^s\right] &= 0 \end{aligned}$$

By bayesian updating one gets:

$$E\left[\hat{a}|\hat{a}^s\right] = \frac{\kappa_s}{\kappa + \kappa_s}\hat{a}^s$$

With this and $\hat{n} = \gamma \hat{a}^s$:

$$\begin{aligned} \left(\left(1-n\right)\left(1-\sigma\right)\alpha - 1\right)\gamma \hat{a}^s + \left(1-n\right)\left(1-\sigma\right)\frac{\kappa_s}{\kappa+\kappa_s}\hat{a}^s &= 0 \\ \left(\left(\left(1-n\right)\left(1-\sigma\right)\alpha - 1\right)\gamma + \left(1-n\right)\left(1-\sigma\right)\frac{\kappa_s}{\kappa+\kappa_s}\right)\hat{a}^s &= 0 \end{aligned}$$

Finally:

$$\gamma = \frac{\left(1-n\right)\left(1-\sigma\right)}{1-\left(1-\sigma\right)\left(1-n\right)\alpha}\frac{\kappa_s}{\kappa+\kappa_s}$$

Note that when the signal becomes arbitrarily good ($\kappa_s \to \infty$) the solution converges to that of perfect information:

$$\lim_{\kappa_s \to \infty} \gamma = \frac{\left(1-\sigma\right)\left(1-n\right)}{1-\left(1-\sigma\right)\left(1-n\right)\alpha}$$

19 A static problem with signals and AR(1) shocks

Now suppose that a follows an AR(1) process so that:

$$\ln a = \rho \ln a_{-} + \epsilon \qquad \epsilon \sim N\left(0, \kappa^{-1}\right)$$

In the perfect information case the solution does not change since the problem that the agent faces is static and the previous the current value of a is known when taking decisions. If there is imperfect information over a one has that the value of a_{-} is known when choosing n, as well as an unbiased signal a^s such that $\ln a^s \sim N\left(\ln a, \kappa_s^{-1}\right)$. After n is chosen the agent observes a and consumes.

Given the information structure the agent enters the period with a prior over a given by:

$$\ln a \sim N\left(\rho \ln a_{-}, k^{-1}\right)$$

This information is updated with the signal to get a posterior distribution over a given by:

$$\ln a \sim N\left(\frac{k\rho \ln a_{-} + k_s \ln a^s}{k + k_s}, (k + k_s)^{-1}\right)$$

The problem is as before:

$$\max \quad E\left[\frac{c^{1-\sigma}}{1-\sigma} + \chi \ln(1-n) \,|a^s, a_{-}\right] \qquad \text{s.t. } c = an^{\alpha}$$

The FOC are:

$$\begin{aligned} -\frac{\chi}{1-n} + E\left[c^{-\sigma}\alpha a n^{\alpha-1} | a^s, a_{-}\right] &= 0 \\ an^{\alpha} - c &= 0 \end{aligned}$$

As before one can linearize these FOC around an arbitrary point and solve them:

$$\begin{aligned} -\frac{\chi}{(1-n)^2} n\hat{n} + c^{-\sigma}\alpha a n^{\alpha-1} E\left[\left(-\sigma\hat{c} + \hat{a} + (\alpha-1)\hat{n}\right)|a^s, a_{-}\right] &\approx 0 \\ \hat{a} + \alpha\hat{n} - \hat{c} &\approx 0 \end{aligned}$$

And by canceling terms:

$$\begin{aligned} \left((\alpha-1) - \frac{n}{1-n}\right)\hat{n} + E\left[\left(-\sigma\hat{c} + \hat{a}\right)|a^s, a_{-}\right] &\approx 0 \\ \hat{a} + \alpha\hat{n} - \hat{c} &\approx 0 \end{aligned}$$

Guess that the solution has the form:

$$\hat{n} = \gamma_1 \hat{a}_{-} + \gamma_2 \hat{a}^s \qquad \hat{c} = \pi_1 \hat{a}_{-} + \pi_2 \hat{a}^s + \pi_3 \hat{a}$$

$$\begin{bmatrix} \hat{n} \\ \hat{c} \end{bmatrix} = \begin{bmatrix} \gamma_1 & \gamma_2 & 0 \\ \pi_1 & \pi_2 & \pi_3 \end{bmatrix} \begin{bmatrix} \hat{a}_{-} \\ \hat{a}^s \\ \hat{a} \end{bmatrix}$$

Using the second equation:

$$\begin{aligned} \hat{a} + \alpha\left(\gamma_1\hat{a}_{-} + \gamma_2\hat{a}^s\right) - \left(\pi_1\hat{a}_{-} + \pi_2\hat{a}^s + \pi_3\hat{a}\right) &= 0 \\ (1-\pi_3)\hat{a} + (\alpha\gamma_1 - \pi_1)\hat{a}_{-} + (\alpha\gamma_2 - \pi_2)\hat{a}^s &= 0 \end{aligned}$$

This equation gives:

$$\pi_1 = \alpha\gamma_1 \qquad \pi_2 = \alpha\gamma_2 \qquad \pi_3 = 1$$

Using the first equation, and replacing consumption:

$$\begin{aligned}
\left((\alpha-1)-\frac{n}{1-n}\right)\hat{n}+E\left[(-\sigma\hat{c}+\hat{a})\,|a^s,a_-\right] &= 0\\
\left((\alpha-1)-\frac{n}{1-n}\right)\hat{n}+E\left[(-\sigma\hat{a}-\sigma\alpha\hat{n}+\hat{a})\,|a^s,a_-\right] &= 0\\
\left((\alpha-1)-\frac{n}{1-n}-\sigma\alpha\right)\hat{n}+(1-\sigma)\,E\left[\hat{a}|a^s,a_-\right] &= 0\\
\left((1-n)(1-\sigma)\alpha-1\right)\hat{n}+(1-n)(1-\sigma)\left(\frac{k\rho}{k+k_s}\hat{a}_-+\frac{k_s}{k+k_s}\hat{a}^s\right) &= 0\\
\Theta\left(\gamma_1\hat{a}_-+\gamma_2\hat{a}^s\right)+(1-n)(1-\sigma)\left(\frac{k\rho}{k+k_s}\hat{a}_-+\frac{k_s}{k+k_s}\hat{a}^s\right) &= 0\\
\left(\Theta\gamma_1+(1-n)(1-\sigma)\frac{k\rho}{k+k_s}\right)\hat{a}_-+\left(\Theta\gamma_2+(1-n)(1-\sigma)\frac{k_s}{k+k_s}\right)\hat{a}^s &= 0
\end{aligned}$$

Finally:

$$\gamma_1=\frac{(1-n)(1-\sigma)}{1-(1-n)(1-\sigma)\alpha}\frac{k\rho}{k+k_s}\qquad \gamma_2=\frac{(1-n)(1-\sigma)}{1-(1-n)(1-\sigma)\alpha}\frac{k_s}{k+k_s}$$

Note that when the signal becomes arbitrarily good ($\kappa_s\to\infty$) the solution converges to that of perfect information:

$$\lim_{\kappa_s\to\infty}\gamma_1=0\qquad \lim_{\kappa_s\to\infty}\gamma_1=\frac{(1-n)(1-\sigma)}{1-(1-n)(1-\sigma)\alpha}$$

20 Incomplete information - General solution

Consider a general problem where x denotes the vector of endogenous state variables, d the vector of control (or decision) variables and z the vector of exogenous state variables. Suppose that:

$$\hat{z}_{t+1} = \rho \hat{z}_t + \epsilon_{t+1}$$

and that the current value of $\hat{z}$ is not observable to the agents. Instead agents observe past values of the exogenous variables $\hat{z}_-$ and a contemporaneous signal z^s. The exogenous states of the solution are then $S = [z_-, z^s]'$.

Assume that $\epsilon \overset{iid}{\sim} N\left(0, \alpha^{-1}\right)$. Then, conditional on z_- one gets $\hat{z} \sim N\left(\rho \hat{z}_-, \alpha^{-1}\right)$, this is the prior distribution of the agents, that is, prior to observing the signal. Letting $\rho = 0$ one gets to the iid case.

Agents receive a (common) signal z^s of the form:

$$\hat{z}^s = \hat{z} + \epsilon^s \qquad \epsilon^s \sim N\left(0, \alpha_s^{-1}\right)$$

such that $\hat{z}^s \sim N\left(\hat{z}, \alpha_s^{-1}\right)$. Then it follows that, given z_- and z^s, the posterior distribution of z is

$$z \sim N\left(\frac{\alpha}{\alpha + \alpha_s}\rho \ln z_- + \frac{\alpha_s}{\alpha + \alpha_s} \ln z^s, (\alpha + \alpha_s)^{-1}\right)$$

when z is a scalar. This formula can be generalized for the multivariate case.

Call $\hat{\alpha} = \frac{\alpha_s}{\alpha + \alpha_s}$ the relative precision of the signal. In the linear model only the expected value of z is relevant.

Note that the system of FOC is obtained as a function of z and not z^s, it can be expressed as:

$$f\left(x_t, d_t, s_t, x_{t+1}, d_{t+1}, s_{t+1}\right) \approx A_1 \begin{bmatrix} \hat{x}_t \\ \hat{d}_t \end{bmatrix} + A_2 E^s \begin{bmatrix} \hat{x}_{t+1} \\ \hat{d}_{t+1} \end{bmatrix} + Z_1 E^s\left[\hat{z}_t\right] + Z_2 E^s\left[\hat{z}_{t+1}\right]$$

The objective is to find laws of motion of the form:

$$\begin{aligned} \hat{x}_{t+1} &= A\hat{x}_t + B\hat{S}_t \\ \hat{d}_t &= C\hat{x}_t + D\hat{S}_t \end{aligned}$$

As before, by certainty equivalence, matrices A and C can be obtained from solving the non-stochastic model, the solution is:

$$A = v_x \Omega_x v_x^{-1} \qquad C = v_{dx} v_x^{-1}$$

where: $\mathcal{A} = -A_2^{-1} A_1$ and V and Ω are given by the Eigen-decomposition of $\mathcal{A} = V \Omega V^{-1}$.

Knowing A and C its possible to find B and D by replacing on the FOC.

The expected value of $\hat{z}$ conditional on $\hat{z}_-$ and $\hat{z}^s$ is a linear function. In this case:

$$E\left[\hat{z} | \hat{z}_-, \hat{z}^s\right] = \frac{\alpha\rho}{\alpha + \alpha_s}\hat{z}_- + \frac{\alpha_s}{\alpha + \alpha_s}\hat{z}^s = \begin{bmatrix} \frac{\alpha\rho}{\alpha+\alpha_s} & \frac{\alpha_s}{\alpha+\alpha_s} \end{bmatrix} \begin{bmatrix} \hat{z}_- \\ \hat{z}^s \end{bmatrix}$$

In what follows $\Lambda_1 = \begin{bmatrix} \frac{\alpha\rho}{\alpha+\alpha_s} & \frac{\alpha_s}{\alpha+\alpha_s} \end{bmatrix}$, this matrix represents the linear operator $E^s[\cdot]$. Finally note that

$$E^s\left[\hat{z}_{t+1}\right] = E^s\left[E\left[\hat{z}_{t+1} | \hat{z}_t\right]\right] = E^s\left[\rho \hat{z}_t\right] = \rho \Lambda_1 \hat{S}_t$$

and that:

$$E^s\left[S_{t+1}\right] = \begin{bmatrix} E^s\left[\hat{z}_t\right] \\ E^s\left[\hat{z}^s_{t+1}\right] \end{bmatrix} = \begin{bmatrix} \Lambda_1 \hat{S}_t \\ E^s\left[E\left[\hat{z}^s_{t+1} | \hat{z}_{t+1}\right]\right] \end{bmatrix} = \begin{bmatrix} \Lambda_1 \hat{S}_t \\ E^s\left[\hat{z}_{t+1}\right] \end{bmatrix} = \begin{bmatrix} \Lambda_1 \hat{S}_t \\ \rho \Lambda_1 \hat{S}_t \end{bmatrix} = \begin{bmatrix} \Lambda_1 \\ \rho \Lambda_1 \end{bmatrix} \hat{S}_t = \Lambda_2 \hat{S}_t$$

Using this one gets:

$$
\begin{aligned}
A_1 \begin{bmatrix} \hat{x}_t \\ \hat{d}_t \end{bmatrix} + A_2 E^s \begin{bmatrix} \hat{x}_{t+1} \\ \hat{d}_{t+1} \end{bmatrix} + Z_1 E^s \left[\hat{z}_t\right] + Z_2 E^s \left[\hat{z}_{t+1}\right] &= 0 \\
A_1 \begin{bmatrix} \hat{x}_t \\ C\hat{x}_t + D\hat{S}_t \end{bmatrix} + A_2 E_t^s \begin{bmatrix} \hat{x}_{t+1} \\ C\hat{x}_{t+1} + D\hat{S}_{t+1} \end{bmatrix} + Z_1 \Lambda_1 \hat{S}_t + Z_2 \rho \Lambda_1 \hat{S}_t &= 0 \\
A_1 \begin{bmatrix} \hat{x}_t \\ C\hat{x}_t + D\hat{S}_t \end{bmatrix} + A_2 \begin{bmatrix} A\hat{x}_t + B\hat{S}_t \\ C\left(A\hat{x}_t + B\hat{S}_t\right) + D\Lambda_2 \hat{S}_t \end{bmatrix} + Z_1 \Lambda_1 \hat{S}_t + Z_2 \rho \Lambda_1 \hat{S}_t &= 0 \\
A_1 \begin{bmatrix} \hat{x}_t \\ C\hat{x}_t + D\hat{S}_t \end{bmatrix} + A_2 \begin{bmatrix} A\hat{x}_t + B\hat{S}_t \\ CA\hat{x}_t + \left(CB + D\Lambda_2\right) \hat{S}_t \end{bmatrix} + \left(Z_1 + Z_2 \rho\right) \Lambda_1 \hat{S}_t &= 0
\end{aligned}
$$

Letting $A_1 = [A_{1x}\, A_{1d}]$ and $A_2 = [A_{2x}\, A_{2d}]$ one has:

$$
\begin{aligned}
A_{1x} x_t + A_{1d} \left(C\hat{x}_t + D\hat{S}_t\right) + A_{2x} \left(A\hat{x}_t + B\hat{S}_t\right) + A_{2d} \left(CA\hat{x}_t + \left(CB + D\Lambda_2\right) \hat{S}_t\right) + \left(Z_1 + Z_2\rho\right) \Lambda_1 \hat{S}_t &= 0 \\
\left(A_{1x} + A_{1d}C + A_{2x}A + A_{2d}CA\right) \hat{x}_t + \left(A_{1d}D + A_{2x}B + A_{2d}\left(CB + D\Lambda_2\right) + \left(Z_1 + Z_2\rho\right)\Lambda_1\right) \hat{S}_t &= 0 \\
\left(A_{1x} + A_{1d}C + A_{2x}A + A_{2d}CA\right) \hat{x}_t + \left(A_{1d}D + \left(A_{2x} + A_{2d}C\right) B + A_{2d}D\Lambda_2 + \left(Z_1 + Z_2\rho\right)\Lambda_1\right) \hat{S}_t &= 0
\end{aligned}
$$

At this point it can be checked that:

$$
A_{1x} + A_{1d}C + A_{2x}A + A_{2d}CA = 0
$$

And then B and D are obtained such that:

$$
A_{1d}D + \left(A_{2x} + A_{2d}C\right) B + A_{2d}D\Lambda_2 + \left(Z_1 + Z_2\rho\right)\Lambda_1 = 0_{3\times n_s}
$$

Vectorizing:

$$
\begin{aligned}
\mathrm{vec}\left(A_{1d}D\right) + \mathrm{vec}\left(\left(A_{2x} + A_{2d}C\right) B\right) + \mathrm{vec}\left(A_{2d}D\Lambda_2\right) + \mathrm{vec}\left(\left(Z_1 + Z_2\rho\right)\Lambda_1\right) &= 0 \\
\left(I_{n_s} \otimes A_{1d}\right) \mathrm{vec}\left(D\right) + \left(I_{n_s} \otimes \left(A_{2x} + A_{2d}C\right)\right) \mathrm{vec}\left(B\right) + \left(\Lambda_2' \otimes A_{2d}\right) \mathrm{vec}\left(D\right) + \mathrm{vec}\left(\left(Z_1 + Z_2\rho\right)\Lambda_1\right) &= 0 \\
\left(\left(I_{n_s} \otimes A_{1d}\right) + \left(\Lambda_2' \otimes A_{2d}\right)\right) \mathrm{vec}\left(D\right) + \left(I_{n_s} \otimes \left(A_{2x} + A_{2d}C\right)\right) \mathrm{vec}\left(B\right) + \mathrm{vec}\left(\left(Z_1 + Z_2\rho\right)\Lambda_1\right) &= 0
\end{aligned}
$$

The system of equations can be stacked to give:

$$
\begin{bmatrix} \left(I_{n_s} \otimes \left(A_{2x} + A_{2d}C\right)\right) & \left(\left(I_{n_s} \otimes A_{1d}\right) + \left(\Lambda_2' \otimes A_{2d}\right)\right) \end{bmatrix} \begin{bmatrix} \mathrm{vec}\left(B\right) \\ \mathrm{vec}\left(D\right) \end{bmatrix} = -\mathrm{vec}\left(\left(Z_1 + Z_2\rho\right)\Lambda_1\right)
$$

$$
\begin{bmatrix} \mathrm{vec}\left(B\right) \\ \mathrm{vec}\left(D\right) \end{bmatrix} = -\begin{bmatrix} \left(I_{n_s} \otimes \left(A_{2x} + A_{2d}C\right)\right) & \left(\left(I_{n_s} \otimes A_{1d}\right) + \left(\Lambda_2' \otimes A_{2d}\right)\right) \end{bmatrix}^{-1} \mathrm{vec}\left(\left(Z_1 + Z_2\rho\right)\Lambda_1\right)
$$

21 Asymmetric information - General solution

Consider now a problem in which some of the decisions are taken with knowledge of the current value of $\hat{z}$ and others are taken as before only with knowledge of $\hat{z}_-$ and $\hat{z}^s$. Since decisions are taken with two different information sets I treat different the coefficients on variables depending on the information set used on them. The relevant exogenous state vector is $S = [z, z_-, z^s]'$.

The system of FOC can be expressed as:

$$0 \approx A_1 \begin{bmatrix} \hat{x}_t \\ \hat{d}_t \end{bmatrix} + A_2 E \begin{bmatrix} \hat{x}_{t+1} \\ \hat{d}_{t+1} \end{bmatrix} + A_3 E^s \begin{bmatrix} \hat{x}_{t+1} \\ \hat{d}_{t+1} \end{bmatrix} + Z_1 \hat{z}_t + Z_2 E^s [\hat{z}_t] + Z_3 E [\hat{z}_{t+1}] + Z_4 E^s [\hat{z}_{t+1}]$$

In this case $Z_3 = Z_4 = 0$ but I include them for completeness of the argument.

The objective is to find laws of motion of the form:

$$\begin{aligned} \hat{x}_{t+1} &= A\hat{x}_t + B\hat{S}_t \\ \hat{d}_t &= C\hat{x}_t + D\hat{S}_t \end{aligned}$$

As before, by certainty equivalence, matrices A and C can be obtained from solving the non-stochastic model, the solution is:

$$A = v_x \Omega_x v_x^{-1} \qquad C = v_{dx} v_x^{-1}$$

where: $\mathcal{A} = -A_2^{-1} A_1$ and V and Ω are given by the Eigen-decomposition of $\mathcal{A} = V\Omega V^{-1}$.

Knowing A and C its possible to find B and D by replacing on the FOC.

The expected value of $\hat{z}$ conditional on $\hat{z}_-$ and $\hat{z}^s$ is a linear function. In this case:

$$E[\hat{z}|\hat{z}_-, \hat{z}^s] = \frac{\alpha\rho}{\alpha + \alpha_s}\hat{z}_- + \frac{\alpha_s}{\alpha + \alpha_s}\hat{z}^s = \begin{bmatrix} 0 & \frac{\alpha\rho}{\alpha+\alpha_s} & \frac{\alpha_s}{\alpha+\alpha_s} \end{bmatrix} \begin{bmatrix} \hat{z} \\ \hat{z}_- \\ \hat{z}^s \end{bmatrix}$$

In what follows $\Lambda_1 = \begin{bmatrix} 0 & \frac{\alpha\rho}{\alpha+\alpha_s} & \frac{\alpha_s}{\alpha+\alpha_s} \end{bmatrix}$, this matrix represents the linear operator $E^s[\cdot]$. Finally note that

$$E^s[\hat{z}_{t+1}] = E^s[E[\hat{z}_{t+1}|\hat{z}_t]] = E^s[\rho\hat{z}_t] = \rho\Lambda_1\hat{S}_t$$

and that:

$$E^s\left[\hat{S}_{t+1}\right] = \begin{bmatrix} E^s[\hat{z}_{t+1}] \\ E^s[\hat{z}_t] \\ E^s[\hat{z}^s_{t+1}] \end{bmatrix} = \begin{bmatrix} E^s[\hat{z}_{t+1}] \\ \Lambda_1\hat{S}_t \\ E^s\left[E\left[\hat{z}^s_{t+1}|\hat{z}_{t+1}\right]\right] \end{bmatrix} = \begin{bmatrix} E^s[\hat{z}_{t+1}] \\ \Lambda_1\hat{S}_t \\ E^s[\hat{z}_{t+1}] \end{bmatrix} = \begin{bmatrix} \rho\Lambda_1\hat{S}_t \\ \Lambda_1\hat{S}_t \\ \rho\Lambda_1\hat{S}_t \end{bmatrix} = \begin{bmatrix} \rho\Lambda_1 \\ \Lambda_1 \\ \rho\Lambda_1 \end{bmatrix}\hat{S}_t = \Lambda_2\hat{S}_t$$

Note that by construction Λ_1 and Λ_2 make the weight on $\hat{z}$ zero.

It is also necessary to define the "complete information" expected value operators:

$$E[z_{t+1}] = \begin{bmatrix} \rho & 0 & 0 \end{bmatrix} S_t = \Lambda_3 S_t$$

$$E\left[\hat{S}_{t+1}\right] = \begin{bmatrix} E[\hat{z}_{t+1}] \\ E[\hat{z}_t] \\ E[\hat{z}^s_{t+1}] \end{bmatrix} = \begin{bmatrix} \rho & 0 & 0 \\ 1 & 0 & 0 \\ \rho & 0 & 0 \end{bmatrix}\hat{S}_t = \Lambda_4\hat{S}_t$$

For what follows let $\Lambda_0 = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$. Using this one gets:

$$\begin{aligned}
A_1 \begin{bmatrix} \hat{x}_t \\ \hat{d}_t \end{bmatrix} + A_2 E \begin{bmatrix} \hat{x}_{t+1} \\ \hat{d}_{t+1} \end{bmatrix} + A_3 E^s \begin{bmatrix} \hat{x}_{t+1} \\ \hat{d}_{t+1} \end{bmatrix} + Z_1 \hat{z}_t + Z_2 E^s \left[\hat{z}_t\right] + Z_3 E\left[\hat{z}_{t+1}\right] + Z_4 E^s \left[\hat{z}_{t+1}\right] &= 0 \\
A_1 \begin{bmatrix} \hat{x}_t \\ \hat{d}_t \end{bmatrix} + A_2 E \begin{bmatrix} \hat{x}_{t+1} \\ \hat{d}_{t+1} \end{bmatrix} + A_3 E^s \begin{bmatrix} \hat{x}_{t+1} \\ \hat{d}_{t+1} \end{bmatrix} + \left[Z_1 \Lambda_0 + Z_2 \Lambda_1 + Z_3 \Lambda_3 + Z_4 \rho \Lambda_1\right] \hat{S}_t &= 0 \\
A_1 \begin{bmatrix} \hat{x}_t \\ C\hat{x}_t + D\hat{S}_t \end{bmatrix} + A_2 E_t \begin{bmatrix} \hat{x}_{t+1} \\ C\hat{x}_{t+1} + D\hat{S}_{t+1} \end{bmatrix} + A_3 E^s \begin{bmatrix} \hat{x}_{t+1} \\ C\hat{x}_{t+1} + D\hat{S}_{t+1} \end{bmatrix} + Z_0 \hat{S}_t &= 0 \\
A_1 \begin{bmatrix} \hat{x}_t \\ C\hat{x}_t + D\hat{S}_t \end{bmatrix} + A_2 \begin{bmatrix} A\hat{x}_t + B\hat{S}_t \\ C\left(A\hat{x}_t + B\hat{S}_t\right) + D\Lambda_4 \hat{S}_t \end{bmatrix} + A_3 \begin{bmatrix} A\hat{x}_t + B\hat{S}_t \\ C\left(A\hat{x}_t + B\hat{S}_t\right) + D\Lambda_2 \hat{S}_t \end{bmatrix} + Z_0 \hat{S}_t &= 0 \\
A_1 \begin{bmatrix} \hat{x}_t \\ C\hat{x}_t + D\hat{S}_t \end{bmatrix} + A_2 \begin{bmatrix} A\hat{x}_t + B\hat{S}_t \\ CA\hat{x}_t + \left(CB + D\Lambda_4\right) \hat{S}_t \end{bmatrix} + A_3 \begin{bmatrix} A\hat{x}_t + B\hat{S}_t \\ CA\hat{x}_t + \left(CB + D\Lambda_2\right) \hat{S}_t \end{bmatrix} + Z_0 \hat{S}_t &= 0
\end{aligned}$$

Where $Z_0 = Z_1 \Lambda_0 + Z_2 \Lambda_1 + Z_3 \Lambda_3 + Z_4 \rho \Lambda_1$.

Letting $A_1 = \left[A_{1x}\, A_{1d}\right]$, $A_2 = \left[A_{2x}\, A_{2d}\right]$ and $A_3 = \left[A_{3x}\, A_{3d}\right]$ one has:

$$\begin{aligned}
A_{1x} x_t + A_{1d}\left(C\hat{x}_t + D\hat{S}_t\right) + A_{2x}\left(A\hat{x}_t + B\hat{S}_t\right) + A_{2d}\left(CA\hat{x}_t + \left(CB + D\Lambda_4\right)\hat{S}_t\right) & \\
+ A_{3x}\left(A\hat{x}_t + B\hat{S}_t\right) + A_{3d}\left(CA\hat{x}_t + \left(CB + D\Lambda_2\right)\hat{S}_t\right) + Z_0 \hat{S}_t &= 0 \\
\left(A_{1x} + A_{1d}C + A_{2x}A + A_{2d}CA + A_{3x}A + A_{3d}CA\right)\hat{x}_t & \\
+ \left(A_{1d}D + A_{2x}B + A_{2d}\left(CB + D\Lambda_4\right) + A_{3x}B + A_{3d}\left(CB + D\Lambda_2\right) + Z_0\right)\hat{S}_t &= 0
\end{aligned}$$

At this point it can be checked that:

$$A_{1x} + A_{1d}C + A_{2x}A + A_{2d}CA + A_{3x}A + A_{3d}CA = 0$$

And then B and D are obtained such that:

$$\left(A_{2x} + A_{3x} + \left(A_{2d} + A_{3d}\right)C\right)B + A_{1d}D + A_{2d}D\Lambda_4 + A_{3d}D\Lambda_2 + Z_0 = 0_{3\times n_s}$$

Vectorizing:

$$\begin{aligned}
\operatorname{vec}\left(\left(A_{2x} + A_{3x} + \left(A_{2d} + A_{3d}\right)C\right)B\right) + \operatorname{vec}\left(A_{1d}D\right) + \operatorname{vec}\left(A_{2d}D\Lambda_4\right) + \operatorname{vec}\left(A_{3d}D\Lambda_2\right) + \operatorname{vec}\left(Z_0\right) &= 0 \\
\left[I_{n_s} \otimes \left(A_{2x} + A_{3x} + \left(A_{2d} + A_{3d}\right)C\right)\right]\operatorname{vec}(B) + \left[I_{n_s} \otimes A_{1d} + \Lambda_4' \otimes A_{2d} + \Lambda_2' \otimes A_{3d}\right]\operatorname{vec}(D) + \operatorname{vec}\left(Z_0\right) &= 0
\end{aligned}$$

The system of equations can be stacked to give:

$$\begin{bmatrix} \left(I_{n_s} \otimes \left(A_{2x} + A_{3x} + \left(A_{2d} + A_{3d}\right)C\right)\right) & \left(I_{n_s} \otimes A_{1d} + \Lambda_4' \otimes A_{2d} + \Lambda_2' \otimes A_{3d}\right) \end{bmatrix} \begin{bmatrix} \operatorname{vec}(B) \\ \operatorname{vec}(D) \end{bmatrix} = -\operatorname{vec}\left(Z_0\right)$$

$$\begin{bmatrix} \operatorname{vec}(B) \\ \operatorname{vec}(D) \end{bmatrix} = -\begin{bmatrix} \left(I_{n_s} \otimes \left(A_{2x} + A_{3x} + \left(A_{2d} + A_{3d}\right)C\right)\right) & \left(I_{n_s} \otimes A_{1d} + \Lambda_4' \otimes A_{2d} + \Lambda_2' \otimes A_{3d}\right) \end{bmatrix}^{-1} \operatorname{vec}\left(Z_0\right)$$