Envelope Theorems...

for arbitrary choice sets

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- The Problem:

$$V(t) = \max_{x \in X(t)} f(x, t)$$

$$X^{\star}(t) = \{x \in X(t) \mid f(x, t) = V(t)\}$$

- What are the properties of the value function V?
  - Under what conditions is V differentiable?
  - If differentiable, what is $V^{'}(t)$?
  - How does V relates to its derivative?
- Envelope theorems give answers to these questions.

Standard Envelope Theorem - MWG

- Let $X(t) = \{x \mid g_1(x,t) = 0 \quad , \quad \ldots \quad , g_k(x,t) = 0\}$
- Let $\lambda_1(t), \ldots, \lambda_k(t)$ be the Lagrange multipliers.
- Let V be differentiable at $t_0 \in [0,1]$ and f and g are differentiable in x and t and the solution $X^\star(t)$ is a differentiable function at t_0.
- Then:

$$V'(t_0) = \frac{\partial f(X^\star(t_0), t_0)}{\partial t} - \sum_{i=1}^{k} \lambda_i \frac{\partial g_i(X^\star(t_0), t_0)}{\partial t}$$

  - If there are no restrictions then: $V'(t_0) = \frac{\partial f(X^\star(t_0), t_0)}{\partial t}$
- The proof follows from manipulating FOC.
  - What are the conditions for V and $X^\star$ to be differentiable?

Beneveniste & Scheinkman (1979) and SLP

- An extension to dynamic programming problems.

$$V(t) = \max_{\underline{x} \in \Pi(t)} \sum_{i=0}^{\infty} f(x_i, x_{t+1}, i) \qquad \text{argmax} = \{g(t,i)\}_{i=0}^{\infty}$$

$$\underline{x} = (x_0, x_1, \ldots) \qquad \Pi(t) = \{\underline{x} \,|\, \forall_i x_{i+1} \in \Gamma(x_i) \quad x_0 = t\}$$

- Assumptions over $\Gamma : X \rightrightarrows X$
  - Γ has a convex graph (stronger than convex valued)
  - Non-empty valued, Compact valued and continuous.
  - $int(\Gamma(x)) \neq \emptyset$ for all $x \in X$
- Assumptions over f
  - Bounded
  - Continuous and differentiable (BS) or continuously differentiable (SLP)
  - Concave (BS) or strictly concave (SLP)

Beneveniste & Scheinkman (1979) and SLP

Theorem: If $t_0 \in int(X)$ and $g(t_0, 0) \in int(\Gamma(t_0))$ then V is differentiable (BS) or continuously differentiable (SLP) at t_0 and:

$$V'(t_0) = f_1(t_0, g(t_0, 0), 0)$$

- The proof relies on the convexity of correspondence Γ and the concavity of f to transfer the differentiability of f to V.
- The key of the proof is the following lemma:

Lemma: Let V be a real valued **concave** function defined on a **convex set** $D \subset \mathbb{R}$. If W is a **concave and differentiable** function in a neighborhood N of $t_0 \in D$ with the property that $W(t_0) = V(t_0)$ and $W(t) \leq V(t)$ for $t \in N$, then V is differentiable at t_0. Moreover $V'(t_0) = W'(t_0)$.

Set up

- Objective:
  - To establish differentiability in the absence of convexity and continuity of the choice sets.
  - Dispense with concavity of value function (when possible).
  - Generalize result to include relation between value function and derivative of objective function.
  - Generalize differentiability results to directional derivatives.
    - Inequality constraints give non-differentiable points when binding constraints change.
    - Inequality constraints may induce non-convexities in choice sets.
- Method:
  - Strengthen continuity of value function to obtain differentiability.

Set up

- **Absolute Continuity:** A function $g : [a, b] \to \mathbb{R}$ is absolutely continuous on $[a, b]$ if:

$$\forall_{\epsilon}\exists_{\delta}\forall_{\{a_k,b_k\}_{k=1}^{n}} \quad \sum_{k=1}^{n} |b_k - a_k| < \delta \longrightarrow \sum_{k=1}^{n} |g(b_k) - g(a_k)| < \epsilon$$

  where $\{a_k, b_k\}_{k=1}^{n}$ is a finite (or countable) system of pairwise disjoint subintervals $(a_k, b_k) \subset [a, b]$.
- **Lemma:** If f is absolutely continuous then f is of bounded variation and has a finite derivative almost everywhere.
- **Theorem (Lebesgue):** $g(x) = g(a) + \int_a^x g'(t)\,dt$ for all $x \in [a, b]$ if and only if g is absolutely continuous in $[a, b]$.

Set up

- **Equicontinuity:** A family of functions on t $\{f(x,t)\}_{x\in X}$ parametrized by x is equicontinuous if:

$$\forall_{\epsilon}\exists_{\delta}\forall_{x} \quad \left|t-t'\right|<\delta \longrightarrow \left|f(x,t)-f\left(x,t'\right)\right|<\epsilon$$

- **Equidifferentiability:** A family of functions on t $\{f(x,t)\}_{x\in X}$ parametrized by x is equidifferentiable at t_0 if:

$$\frac{f\left(x,t^{'}\right)-f(x,t_0)}{t^{'}-t_0}$$

  converges uniformly (across x) as $t^{'}\rightarrow t_0$.

- **Lemma:** If $f_t(x,t_0)$ exists for all x and $\{f_t(x,t)\}_{x\in X}$ is equicontinuous on X, then $\{f(x,t)\}_{x\in X}$ is equidifferentiable on t_0.

Theorem 1 - Envelope Formula

Theorem 1: Take $t_0 \in [0,1]$ and $x^\star \in X^\star(t_0)$. Assume $f_t(x^\star, t_0)$ exists.

- If $t_0 > 0$ and V is left-differentiable then $V'(t_0-) \leq f_t(x^\star, t_0)$.
- If $t_0 < 0$ and V is right-differentiable then $V'(t_0+) \geq f_t(x^\star, t_0)$.
- If $t \in (0,1)$ and V is differentiable then $V'(t_0) = f_t(x^\star, t_0)$.

Proof: Board.

Note: This result holds wherever V is differentiable, it does not require any structure over f other than differentiability over t, it does not require any structure over X.

Note: Implicitly the existence of a solution is required, compactness and continuity are sufficient for this.

Theorem 2 - Differentiability A.E.

Theorem 2:

- Suppose that $f(x, \cdot)$ is absolutely continuous (in t) for all $x \in X$.
  - Then its differentiable wrt t almost everywhere (on $[0, 1]$) and for all $x \in X$.
- Suppose that there exists an integrable function $b : [0, 1] \to \mathbb{R}$ such that $|f_t(x, t)| \leq b(t)$ for all x and almost all t.
- Then V is absolutely continuous.

Corollary:

- Since V is absolutely continuous it is differentiable A.E. and:

$$V(t) = V(0) + \int_0^t V^{'}(t)\, dt$$

- By Theorem 1, if V is differentiable its derivative is given by: $V^{'}(t_0) = f_t(x^\star, t_0)$ when $X^\star(t) \neq \emptyset$. If this holds A.E then:

$$V(t) = V(0) + \int_0^t f_t(x^\star(t), t_0)\, dt \qquad x^\star(t) \in X^\star(t)$$

Theorem 2 - Differentiability A.E.

Proof: Let $t^{'}, t^{''} \in [0,1]$ with $t^{'} < t^{''}$:

$$\begin{aligned}\left|V\left(t^{''}\right) - V\left(t^{'}\right)\right| &\leq \sup_{x\in X}\left|f\left(x,t^{''}\right) - f\left(x,t^{'}\right)\right| \\ &= \sup_{x\in X}\left|\int_{t^{'}}^{t^{''}} f_t\left(x,s\right)ds\right| \leq \int_{t^{'}}^{t^{''}} \sup_{x\in X}\left|f_t\left(x,s\right)\right| ds \\ &\leq \int_{t^{'}}^{t^{''}} b\left(s\right)ds\end{aligned}$$

Note that $b(t) \geq 0$ for all t and that $\int_{t^{'}}^{t^{''}} b(s)\,ds < \infty$, then $\left|V\left(t^{''}\right) - V\left(t^{'}\right)\right|$ is always bounded, one can choose $t^{'}$ and $t^{''}$ close enough so as to satisfy absolute continuity.

Theorem 2 - Differentiability A.E.

Note:

- All assumptions over the choice set are dropped, that is, convexity, continuity and compactness.
- Compactness is only dropped formally since it is still assumed (in the corollary) that there is a solution A.E.
- Concaveness of objective function is dropped.
- New assumptions:
  - Stronger form of continuity.
    - This gives differentiability, before it was imposed.
  - Boundedness of derivative.
    - Derivative is dominated by an integrable function.

Theorem 3 - Directional Differentiability Everywhere

Theorem 3: Let $t_0 \in [0,1]$, if $\{f(x,\cdot)\}_{x\in X}$ is equidifferentiable at t_0 and $\sup_{x\in X} |f_t(x,t_0)| < \infty$, and also $X^\star(t) \neq \emptyset$ for all t, then:

- V is left and right differentiable at t_0 with:

$$V(t_0-) = \lim_{t\to t_0-} f_t(x^\star(t), t_0) \qquad V(t_0+) = \lim_{t\to t_0+} f_t(x^\star(t), t_0)$$

- V is differentiable at t_0 if and only if $f_t(x^\star(t), t_0)$ is continuous at t_0

Proof: In the paper (this one is longer).
Note: There are no restrictions over X.

Mechanism Design I

- There is an agent with payoff function $f(x,t)$ that depends on outcome $x \in Y$ and type $t \in [0,1]$.
- There is a mechanism formed by a message $m \in M$ and an outcome function $h : M \to Y$.
- The agent participates in the mechanism by choosing a message, or equivalently an outcome: $x \in X = \{h(m) | m \in M\} \subset Y$.
- We say that $X^{\star}(t)$ is a choice rule implemented by the mechanism.

If $f(x,t)$ is absolutely continuous on t for all $x \in Y$ and $\sup_{x \in Y} |f_t(x,t)|$ is integrable then:

$$V(t)) = V(0) + \int_0^t f_t(x,s)\, ds$$

Mechanism Design I

- This result has been obtained by Mirrlees (1971), Laffont and Maskin (1980), Fudenberg and Tirole (1991) and Williams (1999).
  - The Mayerson-Satterthwaite theorem and the revenue equivalence theorem can be obtained using the formula.
- Note that the space of actions, and messages is completely arbitrary.
  - Any choice rule can be implemented.
- Previous results requires the choice rule to be piecewise continuously differentiable.
- The absolute integrability of f_t can be relaxed to:
  - Spence-Mirrlees single crossing property.
  - Quasilinear preferences with strictly increasing differences.

Mechanism Design - Example

Fudenberg and Tirole (1991) - Optimal Mechanisms
Principal maximizes expected utility subject to agent's IC and IR
Agent has a type $\theta \in [0, 1]$ unknown to the principal.

$$\max_{x(\cdot),t(\cdot)} E_\theta \left[u_p \left(x \left(\theta\right), t \left(\theta\right), \theta\right)\right]$$

s.t. $u_a \left(x \left(\theta\right), t \left(\theta\right), \theta\right) \geq u_a \left(x \left(\hat{\theta}\right), t \left(\hat{\theta}\right), \hat{\theta}\right) \quad u_a \left(x \left(\theta\right), t \left(\theta\right), \theta\right) \geq \underline{u}$

Under monotonicity the second constraint implies
$u_a \left(x \left(0\right), t \left(0\right), 0\right) = \underline{u} = 0$.
Under quasilinear preferences: $u_a \left(x \left(\theta\right), t \left(\theta\right), \theta\right) = v \left(x, \theta\right) + t$.

Mechanism Design - Example

Define the agent's payoff under type θ as:

$$U_a(\theta) = \max_{\hat{\theta}} u_a\left(x\left(\hat{\theta}\right), t\left(\hat{\theta}\right), \theta\right) = u_a(x(\theta), t(\theta), \theta)$$

Using envelope theorem and results above:

$$\begin{aligned} U_a(\theta) &= U_a(0) + \int_0^\theta \frac{\partial u_a(x(s), t(s), s)}{\partial \theta} ds \\ u_a(x(\theta), t(\theta), \theta) &= \underline{u} + \int_0^\theta \frac{\partial v(x(s), s)}{\partial \theta} ds \end{aligned}$$

- This allows to eliminate the IC constraint by the above expression and monotonicity of $x(\cdot)$.

Mechanism Design II

- Suppose that a mechanism implements choice rule $x^\star$ and gives payoff V.
- The following result characterizes types that maximize payoff.

Let $t_0 \in \text{argmax} V(t)$, if $\{f(x, \cdot)\}_{x \in Y}$ is equidifferentiable and $\sup_{x \in Y} |f_t(x, t_0)| < \infty$ then V is differentiable at t_0 and:

$$V^{'}(t_0) = f_t(x^\star, t_0) = 0$$

Beneveniste & Scheinkman (Again)

Corollary: Suppose X is a convex set in a linear space and $f : X \times [0,1] \to \mathbb{R}$ is a concave function. Let $t_0 \in [0,1]$ and $x^\star \in X^\star(t)$ such that $f_t(x^\star, t_0)$ exists. Then V is differentiable at t_0 and $V'(t_0) = f_t(x^\star, t_0)$

Proof: Take $t', t'' \in [0,1]$ and $\lambda \in (0,1)$, for any $x', x'' \in X$:

$$f(x_\lambda, t_\lambda) \geq \lambda f\left(x', t'\right) + (1-\lambda) f\left(x'', t''\right)$$

Taking sup over x' and x'' and using convexity of X one gets:

$$V(t_\lambda) \geq \lambda V\left(t'\right) + (1-\lambda) V\left(t''\right)$$

Then V is directionally differentiable and $V'(t-) \geq V'(t+)$.
From Theorem 1 one gets: $V'(t-) \leq f_t(x^\star, t) \leq V'(t+)$
Joining inequalities one gets the result.

Note: Set X is arbitrary, in particular let $X = \Pi(t)$ and $f(x,t) = u(F(t) - x_1) + \sum \delta^s u(F(x_s) - x_{s+1})$.

Cont. Functions over Compact sets

One can strengthen the results above under standard continuity and compactness assumptions.

Corollary: If X is compact, f is upper-semicontinuous and $f_t(x,t)$ is continuous on x and t then all assumptions of Theorems 2 and 3 are satisfied. Moreover the *sup* in theorem 3 can be replaced by a *max* and V is differentiable if and only if $\{f_t(x,t) | x \in X^\star(t)\}$ is a singleton.

Note:

- Good behavior of the value function does not depend on good behavior of maximizers. No conditions are imposed over $X^\star(t)$ other than non-emptiness.
- Maximizers might be discontinuous in the parameter but V is absolutely continuous.
- If f is strictly concave then $X^\star(t)$ is a singleton, then V is differentiable everywhere, even at parameters where maximizer is not differentiable.
  - The proof of the envelope theorem from FOC depends on differentiability of maximizers.

Saddle point problems

- These problems allow to study:
  - Nash eq. payoffs of two players zero sum games.
  - Ex post efficient mechanisms.
  - Parametrized constraints (Lagrangians).
- Let X and Y be non-empty sets and $f : X \times Y \times [0,1] \to \mathbb{R}$.
- $(x^\star, y^\star)$ is a saddle point at t if:
  $f(x, y^\star, t) \leq f(x^\star, y^\star, t) \leq f(x^\star, y, t)$
- The saddle set is: $X^\star(t) \times Y^\star(t)$ where:

$$X^\star(t) = \operatorname*{argmax}_x \inf_y f(x, y, t) \qquad Y^\star(t) = \operatorname*{argmin}_y \sup_x f(x, y, t)$$

- The saddle value is:

$$V(t) = \sup_x \inf_y f(x, y, t) = \inf_y \sup_x f(x, y.t)$$

Results

- Under extra assumptions over the topological properties of spaces X and Y one can establish that:

If f is absolutely continuous, $X^{\star}(t) \times Y^{\star}(t) \neq \emptyset$, and $|f_t| < b(t)$, then:

- V is absolutely continuous, hence A.E differentiable.

If in addition X and Y satisfy the second axiom of countability , f_t is continuous on t, and $\{f(x, y, \cdot)\}_{(x,y)\in X\times Y}$ is equidifferentiable, then:

- $V'(t_0) = V(0) + \int_0^t f_t(x^{\star}, y^{\star}, s)\, ds$

Morevoer if X and Y are compact sets

- V is everywhere directionally differentiable with:

$$
\begin{aligned}
V'(t+) &= \max_{x\in X^{\star}(t)} \min_{y\in Y^{\star}(t)} f_t(x, y, t) = \min_{y\in Y^{\star}(t)} \max_{x\in X^{\star}(t)} f_t(x, y, t) \\
V'(t-) &= \min_{x\in X^{\star}(t)} \max_{y\in Y^{\star}(t)} f_t(x, y, t) = \max_{y\in Y^{\star}(t)} \min_{x\in X^{\star}(t)} f_t(x, y, t)
\end{aligned}
$$

Parametrized constraints

Consider the problem:

$$V(t) = \sup_{x \in X; g(x,t) \geq 0} f(x,t) \qquad \text{where } g : X \times [0,1] \to \mathbb{R}^k$$

$$X^\star(t) = \{x \in X \mid f(x,t) = V(t) \quad \wedge \quad g(x,t) \geq 0\}$$

- If X is convex, f and g are concave and $\exists_{x' \in X} g\left(x', t\right) \gg 0$ then the constrained maximization can be represented as a saddle point problem for the Lagrangian:

$$L(x, \lambda, t) = f(x,t) + \sum_{i=1}^{k} \lambda_k g_k(x,t)$$

  - $V(t)$ equals the saddle value of the Lagrangian.
  - $X^\star(t)$ and $\Lambda^\star(t)$ form the saddle set, where:

$$\Lambda^\star(t) = \underset{\lambda \in \mathbb{R}_+^k}{\operatorname{argmin}} \left(\sup_{x \in X} L(x, \lambda, t) \right)$$

Parametrized constraints

Suppose X is convex, f and g are continuous and concave in x and f_t and g_t are continuous in (x, t) and $\exists_{x' \in X} g\left(x^{'}, t\right) \gg 0$, then:

- V is absolutely continuous, hence A.E differentiable.
- $V(t_0) = V(0) + \int_0^t L_t(x^\star(s), y^\star(s), s)\, ds$
- V is everywhere directionally differentiable with:

$$
\begin{aligned}
V^{'}(t+) &= \max_{x \in X^\star(t)} \min_{\lambda \in \Lambda^\star(t)} L_t(x, \lambda, t) = \min_{\lambda \in \Lambda^\star(t)} \max_{x \in X^\star(t)} L_t(x, \lambda, t) \\
V^{'}(t-) &= \min_{x \in X^\star(t)} \max_{\lambda \in \Lambda^\star(t)} L_t(x, \lambda, t) = \max_{\lambda \in \Lambda^\star(t)} \min_{x \in X^\star(t)} L_t(x, \lambda, t)
\end{aligned}
$$