Microeconomic Theory[1]
List of Definitions and Propositions

Sergio Ocampo Díaz

University of Minnesota

Contents

[1]These notes are intended to summarize the main concepts, definitions and results of the first year courses in Microeconomic Theory, proofs are only presented for some of the propositions. The notes draw mostly from the courses and some of the textbooks followed during the year, also from a series of papers on general equilibrium and game theory. The notes on the Microeconomic Theory sequence by Joe Steinberg's were of great help. Finally most of what is done is due to discussion with my classmates, specially Dominic Smith, Keler Marku, Vegard Nygard, Emily Moschini, Weiwen Leung and Jorge Mondragón. Please let me know of any errors that persist in the document.
E-mail: ocamp020@umn.edu.

Part I
Mas-Colell, Whinston & Green

1 Preference and Choice

1.1 Preference Relations

Let $\succeq$ be a preference relation on alternative consumption bundles of consumption set X. From $\succeq$ the following relations are defined:

i. Strict preference relation $\succ$: $x \succ y \iff [(x \succeq y) \wedge \neg (y \succeq x)]$

ii. Indifference relation $\sim$: $x \sim y \iff [(x \succeq y) \wedge (y \succeq x)]$

Definition 1.B.1 (Rational preference - complete and transitive) The preference relation $\succeq$ on X is rational if it satisfies:

i. Completeness: $\forall_{x,y \in X} [(x \succeq y) \vee (y \succeq x)]$

ii. Transitivity: $\forall_{x,y,z \in X} [(x \succeq y) \wedge (y \succeq z) \rightarrow (x \succeq z)]$

Note: Completeness implies reflexiveness (defines as $\forall_{x \in X} x \succeq x$).

Proposition 1.B.1 (Properties of $\succ$ and $\sim$) If $\succeq$ is rational then:

i. $\succ$ is both irreflexive ($x \succ x$ never holds) and transitive ($(x \succ y) \wedge (y \succ z) \rightarrow (x \succ z)$).

ii. $\sim$ is reflexive ($\forall_{x \in X} x \sim x$), transitive ($(x \sim y) \wedge (y \sim z) \rightarrow (x \sim z)$) and symmetric ($(x \sim y) \rightarrow (y \sim x)$).

iii. $(x \succ y) \wedge (y \succeq z) \rightarrow (x \succ y)$

Definition 1.B.2 (Representation of preferences) A function $u : X \rightarrow \mathbb{R}$ is a utility function representing $\succeq$ if $\forall_{x,y \in X} (x \succeq y) \iff (u(x) \geq u(y))$.

Note: Let u represent $\succeq$ and $f : \mathbb{R} \rightarrow \mathbb{R}$ be an strictly increasing function, then $v(x) = f(u(x))$ represents $\succeq$.

Proposition 1.B.2 (Necessity of rationality) If $u(x)$ represents a preference relation $\succeq$ then $\succeq$ is rational. (Proof uses completeness and transitivity of $\geq$ relation in $\mathbb{R}$).

1.2 Choice Rules

Define a choice structure $(\mathcal{B}, C(\cdot))$ is formed by a family of non-empty subsets of the choice X ($\mathcal{B}$) and a correspondence from $\mathcal{B}$ to X ($C(\cdot)$).

Definition 1.C.1 (Weak axiom of revealed preferences) The choice structure $(\mathcal{B}, C(\cdot))$ satisfies the WARP if $\forall_{B\in\mathcal{B}}(x, y \in B) \wedge (x \in C(B)) \to \forall_{B'\in\mathcal{B}}[(x, y \in B') \wedge (y \in C(B')) \to (x \in C(B'))]$.

Definition 1-C.2 (Revealed preference relation) Given a choice structure $(\mathcal{B}, C(\cdot))$ the revealed preference relation $\succeq^\star$ is defined by: $x \succeq^\star y \iff \exists_{B\in\mathcal{B}} x, y \in B \wedge x \in C(B)$.

1.3 The Relationship between Preference Relations and Choice Rules

Let $\mathcal{B} \subseteq 2^X$, and $\succeq$ a rational preference relation. $\succeq$ generates the choice structure $(\mathcal{B}, C^\star(., \succeq))$ where $C^\star(B, \succeq) = \{x \in B | \forall_{y\in B} x \succeq y\}$. It is assumed that $\mathcal{B}$ is such that $\forall_{B\in\mathcal{B}} C^\star(B, \succeq) \neq \emptyset$.

Proposition 1.D.1 (Rational preferences to WARP) Suppose $\succeq$ is rational, then the choice structure $(\mathcal{B}, C(\cdot))$ satisfies the WARP.

Definition 1.D.1 (Rationalization of $C(\cdot)$) Given $(\mathcal{B}, C(\cdot))$ $\succeq$ rationalizes $C(\cdot)$ relative to $\mathcal{B}$ if $\forall_{B\in\mathcal{B}} C(B) = C^\star(B, \succeq)$. That is if $\succeq$ generates the choice structure $(\mathcal{B}, C(\cdot))$.

Note: If a preference relation exists that rationalizes $C(\cdot)$ it need not be unique.

Proposition 1.D.2 (Choice structure to rational preferences) If $(\mathcal{B}, C(\cdot))$ is a choice structure such that satisfies WARP and $\mathcal{B}$ contains all subsets of X with up to three elements, then there is a rational preference relation $\succeq$ that rationalizes $C(\cdot)$ relative to $\mathcal{B}$ $(\forall_{B\in\mathcal{B}} C(B) = C^\star(B, \succeq))$. Furthermore $\succeq$ is unique. (Proof uses the revealed preference relation $\succeq^\star$)

2 Consumer Choice

The choice set X will be a commodity space $X \subseteq \mathbb{R}^L$, an $x \in X$ is called a consumption vector or consumption bundle. For simplicity the consumption set will be restricted to $X = \mathbb{R}_+^L$. This set is convex.

2.1 Competitive Budgets

Consumers are assumed to be price takers and prices $p \in \mathbb{R}^L$ are restricted so that $p \gg 0$. Also to have an amount of wealth given by w.

Definition 2.D.1 (Walrasian budget set) Given prices p and wealth w the set of consumption bundles affordable to the consumer are defined by the Walrasian budget set: $B_{p,w} = \left\{x \in \mathbb{R}_+^L \middle| \langle p, x\rangle \leq w\right\}$.

Note: The set $\left\{x \in \mathbb{R}_+^L \middle| \langle p, x\rangle = w\right\}$ is called budget hyperplane. Also both sets are convex, this property depends on the convexity of X.

2.2 Demand Functions and Comparative Statics

A demand correspondence $x(p, w)$ is the set of chosen consumption bundles for each (p, w) pair.

Definition 2.E.1 (Homogeneity of degree zero) The Walrasian demand correspondence $x(p, w)$ is homogenous of degree zero $(\forall_{\alpha>0} x(\alpha p, \alpha w) = x(p, w))$.

Note: The Walrasian budget set does not change when p and w are scaled: $B_{p,w} = B_{\alpha p, \alpha w}$.

Definition 2.E.2 (Walras' Law) The Walrasian demand correspondence $x(p, w)$ satisfies Walras' law: $\forall_{x \in x(p,w)} \langle p, x\rangle = w$.

Wealth Effects (under $x(p, w)$ singled valued, continuous and differentiable)

Definition (Engle function and wealth expansion path) For a fixed $p = \bar{p}$ the demand correspondence $x(\bar{p}, w)$ as a function of wealth is called the Engle function, and $E_{\bar{p}} = \{x(\bar{p}, w) | w > 0\}$ is the wealth expansion path.

Definition (Wealth effect, normal and inferior goods) For good l in $x(p, w)$ the wealth effect is ${}^{\partial x_l(p,w)}/_{\partial w}$. A good is called normal if ${}^{\partial x_l(p,w)}/_{\partial w} \geq 0$ and inferior if ${}^{\partial x_l(p,w)}/_{\partial w} < 0$. The vector of wealth effects is $D_w x(p, w) \in \mathbb{R}^L$.

Price Effects (under $x(p, w)$ singled valued, continuous and differentiable)

Definition (Price effect and Giffen goods) For good l in $x(p, w)$ the price effect of p_k on the demand for x_l is ${}^{\partial x_l(p,w)}/_{\partial p_k}$. A good is called Giffen if ${}^{\partial x_l(p,w)}/_{\partial p_l} > 0$. The matrix of price effects is $D_p x(p, w)$.

Definition 2.E.1 (Implication of homogeneity) If the Walrasian demand function is homogeneous of degree 0, then $\forall_l \sum_{k=1}^{L} \frac{\partial x_l(p,w)}{\partial p_k} p_k + \frac{\partial x_l(p,w)}{\partial w} w = 0$. In matrix notation $D_p x(p,w)\, p + D_w x(p,w)\, w = 0$. (Proof uses Euler's formula or $x(\alpha p, \alpha w) - x(p,w) = 0$ differentiating against α and evaluating $\alpha = 1$.

Definition (Elasticities) Elasticities of demand with respect to prices and wealth are defined as: $\varepsilon_{lk}(p,w) = \frac{\partial x_l(p,w)}{\partial p_k} \frac{p_k}{x_l(p,w)}$ $\quad \varepsilon_{lw}(p,w) = \frac{\partial x_l(p,w)}{\partial w} \frac{w}{x_l(p,w)}$. Note that the last definition can be expressed as: $\forall_l \sum_{k=1}^{L} \varepsilon_{lk}(p,w) + \varepsilon_{lw}(p,w) = 0$.

Definition 2.E.2 (Implication of Walras' law 1) If the Walrasian demand function satisfies Walras' law, then $\forall_k \sum_{l=1}^{L} p_l \frac{\partial x_l(p,w)}{\partial p_k} + x_k(p,w) = 0$ or in matrix notation $p D_p x(p,w) + x(p,w)' = 0'$. (Proof by differentiating Walras' law against prices).

Note: Total expenditure cannot change in response to a change in prices.

Definition 2.E.3 (Implication of Walras' law 2) If the Walrasian demand function satisfies Walras' law, then $\sum_{l=1}^{L} p_l \frac{\partial x_l(p,w)}{\partial w} = 1$ or in matrix notation $p D_w x(p,w) = 1$. (Proof by differentiating Walras' law against wealth).

Note: Total expenditure mush change by an amount equal to any wealth change.

2.3 The Weak Axiom of Revealed Preference and the Law of Demand

Let the demand correspondence $x(p,w)$ be single valued, homogenous of degree 0 and satisfy Walras' law.

Definition 2.F.1 (WARP for demand function) The demand function $x(p,w)$ satisfies WARP if for any two pairs (p,w) and (p',w') we have:

$$\langle p, x(p',w') \rangle \leq w \wedge x(p',w') \neq x(p,w) \rightarrow \langle p', x(p,w) \rangle > w'$$

this means that both $x(p,w)$ and $x(p',w')$ are affordable at (p,w) but $x(p,w)$ is revealed as preferred, then since $x(p',w') \neq x(p,w)$ it must be that $x(p,w)$ is not affordable at (p',w').

Definition (Slutsky wealth compensation) If a consumer faces prices p, wealth w and chooses $x(p,w)$, the Slutsky wealth compensation is defined as the change in wealth necessary to make $x(p,w)$ affordable at any new price vector p': $\Delta w = \langle \Delta p, x(p,w) \rangle = \langle p', x(p,w) \rangle - \langle p, x(p,w) \rangle$.

Definition (Slutsky compensated price changes) A change in price (p to p') is called Slutsky compensated if it is accompanied by a change in wealth equal to the Slutsky wealth compensation.

Proposition 2.F.1 (Compensated law of demand) The Walrasian demand $x(p,w)$ satisfies WARP if and only if for any compensated price change from (p,w) to $(p',w') = (p', \langle p', x(p,w)\rangle)$ we have $\langle (p'-p), (x(p',w') - x(p,w))\rangle \leq 0$ (with strict inequality when $x(p,w) \neq x(p',w')$.

Definition (Substitution effect and Slutsky matrix) The substitution effect (when $x(p,w)$ is differentiable) between commodities l and k is defined as:

$$s_{lk}(p,w) = \frac{\partial x_l(p,w)}{\partial p_k} + \frac{\partial x_l(p,w)}{\partial w} x_k(p,w)$$

The effect on x_l due to a change in p_k compensating wealth to be able to afford the original bundle, this is the effect due only to changes in relative prices. The matrix

$$S(p,w) = D_p x(p,w) + D_w x(p,w) \, x(p,w)'$$

is called the Slutsky matrix

Proposition 2.F.2 (Slutsky matrix I) If $x(p,w)$ is differentiable, homogenous of degree 0 and satisfies Walras' law and WARP then $S(p,w)$ is negative semidefinite ($\forall_{v \in \Re^L} v' S(p,w) v \leq 0$).

Note: This implies that $s_{ll}(p,w) \leq 0$, the substitution effect of a good to a change in its own price is always non-positive. This **does not** imply that matrix $S(p,w)$ is symmetric.

Note: Satisfying WARP is necessary for S being negative semidefinite but having $x(p,w)$ with a negative semidefinite substitution matrix does not imply that $x(p,w)$ satisfies WARP. The sufficient condition is $vS(p,w) < 0$ for $v \neq \alpha p$.

Proposition 2.F.3 (Slutsky matrix II) The Slutsky matrix satisfies: $pS(p,w) = 0$, $S(p,w)p = 0$. (Proof using definition and Walras' law Def 2.E.1).

Note: It follows that $S(p,w)$ is always singular and cannot be negative definite.

3 Classical Demand Theory

3.1 Preference Relations: Basic Properties

Definition 3.B.1 (Rational preferences) The preference relation $\succeq$ on X is rational if it satisfies:

i. Completeness: $\forall_{x,y\in X}\left[(x\succeq y)\vee(y\succeq x)\right]$

ii. Transitivity: $\forall_{x,y,z\in X}\left[(x\succeq y)\wedge(y\succeq z)\rightarrow(x\succeq z)\right]$

Definition 3.B.2 (Monotonicity and strong monotonicity) The preference relation $\succeq$ on X is monotone if $\forall_{x,y\in X} x\gg y\rightarrow x\succ y$. It is strongly monotone if $\forall_{x,y\in X} x\geq y\wedge x\neq y\rightarrow x\succ y$.

Note: If $\succeq$ is strongly monotone then it is monotone

Definition 3.B.3 (Local non-satiation) The preference relation $\succeq$ on X is locally non-satiated if $\forall_{x\in X}\forall_{\epsilon>0}\exists_{y\in X}\left\|y-x\right\|<\epsilon\wedge y\succ x$.

Note: If $\succeq$ is monotone then it is locally non-satiated. Non-satiated preferences rule out thick indifference curves.

Definition 3.B.4 (Convexity) The preference relation $\succeq$ on X is convex if for every $x\in X$ the upper contour set $U(x)=\{y\in X|y\succeq x\}$ is convex.

Note: Convexity is interpreted as diminishing marginal rates of substitution (it takes larger amounts of a commodity to compensate for successive unit losses in other). It also expresses inclination for diversification. This property depends on the convexity of the choice set X.

Definition 3.B.5 (Strict Convexity) The preference relation $\succeq$ on X is strictly convex if

$$\forall_{x,y,z\in X} y\succeq x\wedge z\succeq x\wedge y\neq z\rightarrow\forall_{\alpha\in(0,1)}\alpha y+(1-\alpha)z\succ x$$

Definition 3.B.6 (Homothetic preferences) The preference relation $\succeq$ on X is homothetic if $x\sim y\rightarrow\forall_{\alpha\geq 0}\alpha x\sim\alpha y$. All indifference sets are related by proportional expansion along rays.

Definition 3.B.6 (Quasilinear preferences) The preference relation $\succeq$ on $(-\infty,\infty)\times\Re_+^{L-1}$ is quasilinear with respect to commodity 1 if:

i. All indifference sets are parallel displacements of each other along the axis of commodity 1:

$$x\sim y\rightarrow\forall_{\alpha\in\Re}(x+\alpha e_1)\sim(y+\alpha e_1)$$

ii. Good 1 is desirable: $\forall_{\alpha>0}\forall_x(x+\alpha e_1)\succ x$

Note: One can deduce the consumer's entire preference relation from a single indifference set if preferences are homothetic or quasilinear.

3.2 Preference and Utility

Definition 3.C.1 (Continuity of Preferences)

Sequential Definition The preference relation $\succeq$ on X is continuous if for any sequences of pairs $\{(x_n, y_n)\}$ with $\forall_n x_n \succeq y_n$, $x_n \to x$ and $y_n \to y$ we have $x \succeq y$.

Set Definition The preference relation $\succeq$ on X is continuous if for all $x \in X$ the upper contour $(U(x) = \{y \in X | y \succeq x\})$ and lower contour $(L(x) = \{y \in X | x \succeq y\})$ are closed.

Proposition (Equivalence of Continuity Definitions) Preferences $\succeq$ are continuous in the sequential sense if and only if they are continuous in the set sense.

Proof

i. **Sequential definition implies set definition:**

(a) Let $y \in X$ and x_n be a sequence in $U(y)$ for all n and y_n be defined as the constant sequence equal to y. This implies $x_n \succeq y_n$ for all n. Then if $x_n \to x$ then $x \succeq y$ which means $x \in U(y)$, then $U(y)$ is closed for all y.

(b) Similarly, for $L(y)$ let $y \in X$ and x_n be in $L(y)$ for all n. Then define y_n as the constant sequence equal to y. This implies $x_n \preceq y_n$ for all n. Then if $x_n \to x$ one gets $x \preceq y$ which means $x \in L(y)$, then $L(y)$ is closed for all y.

ii. **Set definition implies sequential definition:**

Let $x_n \to x$ and $y_n \to y$ such that $x_n \succeq y_n$ for all n. Suppose for contradiction that $y \succ x$.

(a) Since $L(x)$ and $U(y)$ are closed it follows that $L^c(x)$ and $U^c(y)$ are open. Note that $y \in L^c(x)$ and $x \in U^c(y)$. Then there exists ϵ such that $\forall_{x' \in B_\epsilon(x)} x' \in L^c(y)$ and $\forall_{y' \in B_\epsilon(y)} y' \in U^c(x)$. Then there exists N such that for $n \geq N$ $x_n \in L^c(y)$ and $y_n \in U^c(x)$ which is $x_n \prec y$ and $y_n \succ x$.

(b) Fix $n \geq N$. Using the relations, since $y \in L^c(x_n)$ and $x \in U^c(y)$ it follows, as before, that there exits M such that for $m \geq M$ $x_n \prec y_m$ and $y_n \succ x_m$.

(c) Joining $x_n \prec y_m \preceq x_m \prec y_n$ which implies by transitivity $x_n \prec y_n$. This is a contradiction.

Proposition 3.C.1 (Utility function) Suppose that the rational relation $\succeq$ on X is continuous, then there is a continuous utility function $u(x)$ that represents $\succeq$. (The proof below uses also strong monotonicity of preferences).

Note: There is not a unique utility function that represents $\succeq$ and not all the functions that represent it are continuous.

Further assumptions

i. If preference relation $\succeq$ is also assumed to be monotone then the utility that represents it is also increasing. $x \gg y \rightarrow u(x) > u(y)$.

ii. If the preference relation $\succeq$ is also assumed to be convex then the utility that represents it is also quasi-concave. The set $\left\{y \in \mathbb{R}_{+}^{L} | u(y) \geq u(x)\right\}$ is convex for all x, or equivalently $\forall_{\alpha \in (0,1)} u(\alpha x + (1-\alpha) y) \geq \min\{u(x), u(y)\}$.

iii. A continuous preference relation $\succeq$ is homothetic if and only if it admits a utility function $u(x)$ that is homogenous of degree one.

iv. A continuous preference relation $\succeq$ is quasi-linear if and only if it admits a utility function $u(x)$ of the form $u(x) = x_1 + \phi(x_2, \ldots, x_L)$.

Proof:

i. Let $\succeq$ be a continuous, monotone and complete preorder on $\mathbb{R}_{+}^{l}$, and $Z = \left\{x \in \mathbb{R}_{+}^{l} | \exists_{\lambda \geq 0} x = \lambda e\right\}$ with $e = (1, \ldots, 1)$.

ii. Let $x \in \mathbb{R}_{+}^{l}$, there exists a unique value $\alpha(x) \geq 0$ such that $x \sim \alpha(x) e$:

   (a) Existence is given in the following way: let $U(x) = \left\{x' \in \mathbb{R}_{+}^{l} | x' \succeq x\right\}$ and $L(x) = \left\{x' \in \mathbb{R}_{+}^{l} | x \succeq x'\right\}$ and define $A_1 = Z \cap U(x)$ and $A_2 = Z \cap L(x)$.

      i. Since Z is closed and, by continuity, $U(x)$ and $L(x)$ are closed, it follows that A_1 and A_2 are closed.

      ii. Note that $A_1 \cup A_2 = Z$, which is a connected set. A connected set cannot be "separated" into two closed sets.

      iii. $A_1 \cap A_2 = \{\alpha \geq 0 | \alpha e \sim x\} \neq \emptyset$ because of the two last points.

   (b) Uniqueness is given since $\alpha > \alpha'$ imply $\alpha e \gg \alpha' e$ and then by monotonicity $\alpha e \succ \alpha' e$.

iii. Define the utility function $u(x) = \alpha(x)$. This utility function represents the preferences.

   (a) Let $x, y \in \mathbb{R}_{+}^{l}$ such that $x \succeq y$, then by construction and transitivity $\alpha(x) e \succeq \alpha(y) e$, by monotonicity of $\succeq$ it must be that $\alpha(x) \geq \alpha(y)$ (otherwise $\alpha(y) e \succ \alpha(x) e$).

   (b) Let $x, y \in \mathbb{R}_{+}^{l}$ such that $\alpha(x) \geq \alpha(y)$, then by monotonicity $\alpha(x) e \succeq \alpha(y) e$, by construction and transitivity $x \succeq y$.

iv. The function $u(x)$ is continuous. For proving this is enough to show that for any x and sequence $\{x_n\}$ such that $x_n \rightarrow x$ then $\alpha(x_n) \rightarrow \alpha(x)$. Suppose it does not.

   (a) First note that, wlog, $\{\alpha(x_n)\}$ is bounded, since $\{x_n\}$ can always be taken to be bounded (if it is not, and since it converges to x, take a subsequence such that $\|x_n - x\| \leq 1$).

(b) Since $\{\alpha(x_n)\}$ is bounded it has a convergent subsequence, since it doesn't converge to α we can take the subsequence such that $\alpha_{n_k} \to \alpha' \neq \alpha(x)$. Wlog assume that $\alpha' > \alpha(x)$.

(c) Choose $\hat{\alpha} \in (\alpha(x), \alpha')$, since $\alpha_{n_k} \to \alpha'$ there is a high enough $\hat{k}$ for which $\forall_{k>\hat{k}} \hat{\alpha} < \alpha(x_{n_k})$, then $\forall_{k>\hat{k}} \hat{\alpha} e \prec x_{n_k}$.

(d) By continuity of preferences (and since $x_{n_k} \to x$) it follows that $\hat{\alpha} e \prec x \sim \alpha(x) e$ which contradicts monotonicity since $\hat{\alpha} > \alpha(x)$. Then it must be that $\alpha(x_n) \to \alpha(x)$

v. Alternatively a function $u : \mathbb{R}^l_+ \to \mathbb{R}$ is continuous if the pre-images of closed sets are closed. WLOG let $[\alpha_1, \alpha_2] \subset \mathbb{R}$ be a closed set, its pre-image is:

$$u^{-1}([\alpha_1, \alpha_2]) = u^{-1}([\alpha_1, \infty]) \cap u^{-1}([0, \alpha_2])$$

where:

$$u^{-1}([\alpha_1, \infty]) = \left\{x \in \mathbb{R}^+_l : u(x) \geq \alpha_1\right\} = \left\{x \in \mathbb{R}^+_l : x \geq \alpha_1 e\right\} = U(\alpha_1 e)$$
$$u^{-1}([0, \alpha_2]) = \left\{x \in \mathbb{R}^+_l : u(x) \leq \alpha_2\right\} = \left\{x \in \mathbb{R}^+_l : x \leq \alpha_2 e\right\} = L(\alpha_2 e)$$

Both sets are closed by continuity of preferences and hence their intersection is also closed. The closed sets in $\mathbb{R}$ are constructed from this.

3.3 The Utility Maximization Problem

The consumer's problem to choose the most preferred bundle given prices $p \gg 0$ and wealth $w > 0$ is states as the utility maximization problem (UMP):

$$\max_{x \geq 0} u(x) \qquad \text{st. } \langle p, x \rangle \leq w$$

Proposition 3.D.1 (Existence of a solution) If $p \gg 0$ and $u(\cdot)$ is continuous, then the UMP has a solution since the budget set $B_{p,w} = \left\{x \in \mathbb{R}^L_+ | \langle p, x \rangle \leq w\right\}$ is compact.

Proposition 3.D.2 (Properties of demand correspondence) Suppose $u(\cdot)$ is a continuous utility function representing a locally non-satiated preference relation $\succeq$. Then the Walrasian demand correspondence $x(p, w)$ possesses the following properties:

i. Homogeneity of degree 0 in (p, w): $x(\alpha p, \alpha w) = x(p, w)$.

ii. Walras' law: $\forall_{x \in x(p,w)} \langle p, x \rangle = w$.

iii. Convexity/Uniqueness: If $\succeq$ is convex (so that $u(\cdot)$ is quasi-concave) then $x(p, w)$ is convex valued. Moreover if $\succeq$ is strictly convex (so that $u(\cdot)$ is strictly quasi-concave) then $x(p, w)$ is single valued.

Definition (Indirect utility function) For each $(p, w) \gg 0$ the utility value of the UMP (the indirect utility function) is denoted $v(p, w) \in \mathbb{R}$, and is equal to $u(x^\star)$ with $x^\star \in x(p, w)$.

Proposition 3.D.3 (Properties of the indirect utility function)

i. Homogenous of degree zero.

ii. Strictly increasing in w and non-increasing in p_l for $l \in \{1, \ldots, L\}$.

iii. Quasi-convex: for all $\bar{v}$ the set $\{(p, w) \,|v(p, w) \leq \bar{v}\}$ is convex.

iv. Continuous in p and w.

Note: The third property **doesn'**t require $u(\cdot)$ to be quasi-concave. Also, if $v(p, w)$ is the indirect utility function associated to utility function $u(\cdot)$, then $\tilde{v}(p, w) = f(v(p, w))$ is the indirect utility function associated to utility function $\tilde{u}(x) = f(u(x))$.

3.4 The Expenditure Minimization Problem

The expenditure minimization problem (EMP) is defines for $p \gg 0$ and $u > u(0)$ as:

$$\min_{x \geq 0} \langle p, x \rangle \qquad \text{s.t. } u(x) \geq u$$

$u(\cdot)$ is assumed to be a continuous function that represents a locally non-satiated preference relation $\succeq$.

Note: For the solution of this problem to exist suffices that the constraint set is non-empty.

Proposition 3.E.1 (Equivalence between UMP and EMP)

i. If $x^\star$ is optimal in the UMP when wealth is $w > 0$, then $x^\star$ is optimal in the EMP when the required utility level is $u(x^\star)$. Moreover the minimized expenditure level is exactly w.

ii. If $x^\star$ is optimal in the EMP when the required utility is $u > u(0)$, then $x^\star$ is optimal in the UMP when wealth is $\langle p, x^\star \rangle$. Moreover the maximized utility level is exactly u.

Definition (Expenditure function) For each $p \gg 0$ and $u > u(0)$ the value of the EMP is denoted $e(p, u)$, and is equal to $\langle p, x^\star \rangle$ with $x^\star$ is any solution to the EMP.

Proposition 3.E.2 (Properties of the expenditure function) Suppose $u(\cdot)$ is a continuous utility function representing a locally non-satiated preference relation $\succeq$. Then the expenditure function $e(p, u)$ is:

i. Homogenous of degree 1 in p.

ii. Strictly increasing in u and non-decreasing in p_l for $l \in \{1, \ldots, L\}$.

iii. Concave in p.

iv. Continuous in p and u.

Definition (Hicksian or compensated demand correspondence) The set of optimal commodity vectors in the EMP is denoted $h(p,u) \subseteq \mathbb{R}^L_+$ is known as the Hicksian demand correspondence.

Proposition 3.E.3 (Properties of the Hicksian demand correspondence)

i. Homogeneity of degree 0 in p: $h(\alpha p, u) = h(p,u)$.

ii. No excess utility: $x \in h(p,u) \rightarrow u(x) = u$.

iii. Convexity/Uniqueness: if $\succeq$ is convex, then $h(p,u)$ is convex valued, and if $\succeq$ is strictly convex, then $h(p,u)$ is also single valued.

Proposition (Equivalence UMP and EMP) For any $p \gg 0$, $w > 0$ and $u > u(0)$ we have by proposition 3.E.1:

$$e(p, v(p,w)) = w \qquad \text{and} \qquad v(p, e(p,u)) = u$$

$$h(p,u) = x(p, e(p,u)) \qquad \text{and} \qquad x(p,w) = h(p, v(p,w))$$

note that the first equality defines the Hicksian demand as the level of demand that would arise if consumer's wealth were simultaneously adjusted to keep her utility level at u. Hence the term compensated.

Proposition 3.E.4 (Compensated law of demand) Suppose $u(\cdot)$ is a continuous utility function representing a locally non-satiated preference relation $\succeq$, and that $h(p,u)$ is single valued for $p \gg 0$. Then the Hicksian demand function satisfies the compensated law of demand:

$$\langle (p' - p), (h(p',u) - h(p,u)) \rangle \leq 0$$

Note: For compensated demand own-price effects are non-positive. If p_l changes it implies $\left(p_l' - p_l\right)[h_l(p',u) - h_l(p,u)] \leq 0$. Also Walrasian demand need not satisfy the law of demand.

3.5 Duality: A Mathematical Introduction

Definition (Half-space and Hyperplane) A half-space is a set of the form $\left\{x \in \mathbb{R}^L | \langle p, x \rangle \geq c\right\}$ for some $p \in \mathbb{R}^L \backslash \{0\}$ and some $c \in \mathbb{R}$. Its boundary is called a hyperplane $\left\{x \in \mathbb{R}^L | \langle p, x \rangle = c\right\}$. Both half-spaces and hyperplanes are convex sets.

Note: The separating hyperplane theorem establishes that for a convex and closed set $K \subset \mathbb{R}^L$ and a point $x \notin K$ there exists a half-space containing K and excluding x.

Definition 3.F.1 (Support function) For any non-empty closed set $K \subset \mathbb{R}^L$, the support function of K is defined for any $p \in \mathbb{R}^L$ as:

$$\mu_K(p) = \inf \{\langle p, x\rangle \,|\, x \in K\}$$

This function is homogenous of degree one and concave.

Note: When K is convex μ_K provides a dual representation of K since for every p the set $\left\{x \in \mathbb{R}^L \middle| \langle p, x\rangle \geq \mu_K(p)\right\}$ is a half space containing K, and for any $x \notin K$ there exists a p such that $\langle p, x\rangle < \mu_K(p)$. Then $K = \left\{x \in \mathbb{R}^L \middle| \forall_{p \in \mathbb{R}^L} \langle p, x\rangle \geq \mu_K(p)\right\}$.

Proposition 3.F.1 (Duality theorem) Let K be non-empty closed set and $\mu_K(\cdot)$ its support function. Then there is a unique $\bar{x} \in K$ such that $\langle \bar{p}, \bar{x}\rangle = \mu_K(\bar{p})$ if and only if $\mu_K(\cdot)$ is differentiable at $\bar{p}$. Moreover, in this case, $\nabla \mu_K(\bar{p}) = \bar{x}$.

3.6 Relationships between Demand, Indirect Utility and Expenditure Functions

It is assumed that $u(\cdot)$ is a continuous utility function representing the locally non-satiated preferences $\succeq$ and that $p \gg 0$. For simplicity it is also assumed that $\succeq$ is convex so that $x(p, w)$ and $h(p, u)$ are single valued.

Proposition 3.G.1 (Hicksian demands from expenditure function) The hicksian demands $h(p, u)$ can be obtained as $h(p, u) = \nabla_p e(p, u)$, this is $h_l(p, u) = {}^{\partial e(p,u)}/_{\partial p_l}$.

Proposition 3.G.2 (Properties of Hicksian demands) Suppose that $h(., u)$ is continuously differentiable at (p, u), and denote its derivate matrix by $D_p h(p, u)$. Then:

i. $D_p h(p, u) = D_p^2 e(p, u)$.

ii. $D_p h(p, u)$ is negative semidefinite (since e is concave).

iii. $D_p h(p, u)$ is symmetric (since e is concave).

iv. $D_p h(p, u)\, p = 0$ since $h(p, u)$ is homogenous of degree 0 in p.

Note: Since ${}^{\partial h_l(p,u)}/_{\partial p_l} \leq 0$ the last property implies that every good needs to have at least one substitute.

Proposition 3.G.3 (Slutsky equation) For all (p, w) and $u = v(p, w)$ we have:

$$\frac{\partial h_l(p, u)}{\partial p_k} = \frac{\partial x_l(p, w)}{\partial p_k} + \frac{\partial x_l(p, w)}{\partial w} x_l(p, w) \qquad D_p h(p, u) = D_p x(p, w) + D_w x(p, w)\, x(p, w)'$$

Note: This proposition implies that $D_p h(p, u) = S(p, w)$ the Slutsky substitution matrix. This matrix is then symmetric and negative semidefinite.

Proposition 3.G.4 (Roy's identity) Suppose that the indirect utility function is differentiable at $(\bar{p}, \bar{w}) \gg 0$. Then

$$x(\bar{p}, \bar{w}) = -\frac{1}{\nabla_w v(\bar{p}, \bar{w})} \nabla_p v(\bar{p}, \bar{w}) \qquad x_l(\bar{p}, \bar{w}) = -\frac{\partial v(\bar{p},\bar{w}) / \partial p_l}{\partial v(\bar{p},\bar{w}) / \partial w}$$

3.7 Integrability

Proposition (Implication of classical demand theory) If a continuously differentiable demand function $x(p, w)$ is generated by rational preferences then it must be homogenous of degree 0, satisfy Walras' law, and have a substitution matrix $S(p, w)$ symmetric and negative semidefinite.

Proposition 3.H.1 (Rationalization of expenditure function) Suppose $e(p, u)$ is strictly increasing in u and is continuous, increasing , homogenous of degree 1, concave and differentiable in p. Then for every utility level u, $e(p, u)$ is the expenditure function associated with the at-least-as-good set

$$V_u = \left\{ x \in \mathbb{R}_+^L | \forall_{p \gg 0} \langle p, x \rangle \geq e(p, u) \right\}$$

With V_u for several values of u one can define a preference relation $\succeq$ that has $e(p, u)$ as its expenditure function.

Note: The differentiability assumption on p is not necessary.

Proposition (Conditions for integrability) The necessary and sufficient conditions for recovering the expenditure function from the walrasian demands is the symmetry and negative semi-definiteness of the Slutsky matrix.

3.8 The Strong Axiom of Revealed Preference

Definition 3.J.1 (Strong axiom of revealed preference) A market demand function $x(p, w)$ satisfies the SARP if for any list $\{(p_1, w_1), \ldots, (p_N, w_N)\}$ with $x(p_n, w_n) \neq x(p_{n+1}, w_{n+1})$ we have $\langle p_N, x(p_1, w_1) \rangle > w_N$ whenever $\langle p_n, x(p_{n+1}, w_{n+1}) \rangle \leq w_n$ for $n \leq N - 1$.

This means that if $x(p_1, w_1)$ is directly or indirectly revealed preferred to $x(p_N, w_N)$, then $x(p_N, w_N)$ cannot be directly revealed preferred to $x(p_1, w_1)$. ($x(p_1, w_1)$ cannot be affordable at (p_n, w_N)).

Proposition 3.J.1 (Rationalization of preferences) If the Walrasian demand function $x(p, w)$ satisfies SARP then there is a rational preference relation $\succeq$ that rationalizes $x(p, w)$. This a preference such that for all (p, w), $x(p, w) \succ y$ for $y \neq x(p, w)$ and $y \in B_{p,w}$.

5 Production

5.1 Production Sets

Definition (Production plans, production sets and transformation function) A production vector or production plan is a vector $y \in \mathbb{R}^L$ that describes the net outputs of the L commodities. Positive numbers denote outputs and negative numbers denote inputs. The production set is denoted as $Y \subset \mathbb{R}^L$. Any possible production plan satisfies $y \in Y$. The transformation property $F(y)$ has the property that: $Y = \left\{y \in \mathbb{R}^L | F(y) \leq 0\right\}$ and $F(y) = 0$ if and only if y is in the boundary of Y. $\left\{y \in \mathbb{R}^L | F(y) \leq 0\right\}$ is known as the transformation frontier.

Definition (Properties of production sets)

i. $Y \neq \emptyset$

ii. Y is closed: $\forall_{\{y_n\} \subset Y} (y_n \rightarrow y) \rightarrow y \in Y$

iii. No free lunch: $y \in Y \wedge y \geq 0 \rightarrow y = 0$

iv. Possibility of inaction: $0 \in Y$

v. Free disposal: $Y - \mathbb{R}^L_+ \subset Y$

vi. Irreversibility: $y \in Y \wedge y \neq 0 \rightarrow -y \notin Y$

vii. Non-increasing returns to scale: $\forall_{y \in Y} \forall_{\alpha \in [0,1]} \alpha y \in Y$

viii. Non-decreasing returns to scale: $\forall_{y \in Y} \forall_{\alpha \geq 1} \alpha y \in Y$

ix. Constant returns to scale: $\forall_{y \in Y} \forall_{\alpha \geq 0} \alpha y \in Y$ (Y is a cone)

x. Additivity (or free entry): $Y + Y \subset Y$ or $\forall_{y,y' \in Y} y + y' \in Y$

xi. Y is convex

xii. Y is a convex cone: $\forall_{y,y' \in Y} \forall_{\alpha,\beta \geq 0} \alpha y + \beta y' \in Y$

Proposition 5.B.1 (Property of production set) If Y is additive and has constant returns to scale if and only if it is a convex cone.

Proposition 5.B.2 (Entrepreneurial factor) For any convex set $Y \subset \mathbb{R}^L$ with $0 \in Y$ there is a constant returns, convex production set $Y' \subset \mathbb{R}^{L+1}$ such that $Y = \left\{y \in \mathbb{R}^L | (y, -1) \in Y'\right\}$. (This set is $Y = \left\{y' \in \mathbb{R}^{L+1} | y' = \alpha(y, -1) \text{ for some } y \in Y \text{and } \alpha \geq 0\right\}$).

5.2 Profit Maximization and Cost Minimization

It is assumed that $p \gg 0$ and that production set Y is non-empty, close and satisfies free disposal.

Definition (Profit Maximization Problem) The profit maximization problem PMP is defined as:

$$\pi(p) = \max_y \langle p, y \rangle \qquad \text{s.t. } y \in Y$$

The supply correspondence is defined as $y(p) = \{y \in Y | \langle p, y \rangle = \pi(p)\}$.

Note: The profit function can be seen as $\pi(p) = -\mu_{-Y}(p)$ where $\mu_{-Y}(\cdot)$ is the support function of $-Y$.

Note: For a single output technology first order conditions are necessary and sufficient for profit maximization if the production set Y is convex.

Proposition 5.C.1 (Properties of PMP) For Y closed that satisfies free disposal:

i. $\pi(\cdot)$ is homogenous of degree 1.

ii. $\pi(\cdot)$ is convex.

iii. If Y is convex, then $Y = \left\{y \in \mathbb{R}^L | \forall_{p \gg 0} \langle p, y \rangle \leq \pi(p)\right\}$.

iv. $y(p)$ is homogenous of degree 0.

v. If Y is convex, then $y(\cdot)$ is convex valued. Moreover if Y is strictly convex, then $y(\cdot)$ is single valued (if non-empty).

vi. **(Hotteling's Lemma)** if $y(\bar{p})$ is single valued, then $\pi(\bar{p})$ is differentiable at $\bar{p}$ and $\nabla \pi(\bar{p}) = y(\bar{p})$.

vii. If $y(\bar{p})$ is differentiable then $Dy(p) = D^2\pi(p)$ is a symmetric and positive semidefinite matrix with $Dy(p)\,p = 0$.

Note: If there is a single output technology, and the production function is homogenous of degree 1 then $y(p)$ is not single valued at any p, thus Hotteling's Lemma is inapplicable.

Definition (Law of supply) If the price of an output rises the supply of the output also rises. If the price of an input increases the demand for that input decreases. This law holds for any change since there is no need for compensation.
In general the law of supply can be viewed as: $\langle (p - p'), (y(p) - y(p')) \rangle \geq 0$.
If differentiable, Property vii establishes the law of supply through the positive semi-definiteness of $Dy(p)$.

Definition (Cost Minimization Problem) For a single output technology where z is a non-negative vector of inputs, $f(z)$ is the production function, q the amount of output and $w \gg 0$ the vector of input prices. The CMP is defined as:

$$c(w,q) = \min_{z \geq 0} \langle w, z \rangle \qquad \text{s.t. } f(z) \geq q$$

where $c(w,q)$ is the cost function and the conditional factor demand correspondence is $z(w,q)$.

Note: Cost minimization is a necessary condition for profit maximization.

Proposition 5.C.2 (Properties of CMP) For Y closed that satisfies free disposal:

i. $c(\cdot)$ is homogeneous of degree 1 in w and nondecreasing in q.

ii. $c(\cdot)$ is concave in w.

iii. If the sets $\{z \geq 0 | f(z) \geq q\}$ are convex for every q, then $Y = \{(-z, q) | \forall_{w \gg 0} \langle w, z \rangle \geq c(w,q)\}$.

iv. $z(\cdot)$ is homogenous of degree zero in w.

v. If the sets $\{z \geq 0 | f(z) \geq q\}$ are convex, then $z(w,q)$ is convex valued. Moreover if If the sets $\{z \geq 0 | f(z) \geq q\}$ are strictly convex, then $z(w,q)$ is single valued.

vi. **(Shepard's Lemma)** if $z(\bar{w}, q)$ is single valued, then $c(\cdot)$ is differentiable with respect to w at $\bar{w}$ and $z(\bar{w}, q) = \nabla_w c(\bar{w}, q)$.

vii. If $z(w,q)$ is differentiable at w, then $D_w z(w,q) = D_w^2 c(w,q)$ is a symmetric and negative semi-definite matrix with $D_w z(w,q) w = 0$.

viii. If $f(\cdot)$ is homogenous of degree 1, then $c(w,q)$ and $z(w,q)$ are homogenous of degree one in q.

ix. If $f(\cdot)$ is concave, then $c(w,q)$ is convex in q.

Note: If there is a single output technology, and the production function is homogenous of degree 1 then $z(p,q)$ can still be single valued, unlike $y(p)$.

Note: Under convexity there is a one-to-one correspondence between profit and cost function, since the production set can be obtained from both.

Definition (Profit Maximization Problem II) Using the cost function the PMP becomes:

$$\pi(p) = \max_{q \geq 0} pq - c(w,q)$$

5.3 Efficient Production

Definition 5.F.1 (Efficient production plan) A production vector $y \in Y$ is efficient if there is no $y' \in Y$ such that $y' \geq y$ and $y \neq y$.

Note: All efficient production plans are in the boundary of Y but not all boundary points are efficient.

Proposition 5.F.1 (Efficiency and PMP) If y is profit maximizing for some $p \gg 0$, then y is efficient.

Note: This is true even for non-convex Y. Prices must be strictly positive.

Proposition 5.F.1 (Efficiency and PMP II) Suppose Y is convex. If $y \in Y$ is efficient then there exists $p \geq 0$ for which y is profit maximizing. (Proof with separating hyperplane theorem).

Note: This proposition cannot be strengthened so that $p \gg 0$.

Part II

Jan Werner

6 Rationalizability in Production

Definition (Profit Rationalization) Let $\pi : \mathbb{R}^L \to \mathbb{R}$ be a profit function such that $\pi(p)$ indicates the maximum profit achieved under prices $p \in \mathbb{R}^L$. A production set Y profit rationalizes π if $\pi(p) = \max\{p \cdot y | y \in Y\}$.

Proposition (Rationalizability of Profit Function) If π is homogeneous of degree 1, convex, and lower semi-continuous, then there exists a closed and convex set Y that profit-rationalizes π. The set is: $Y = \left\{y \in \mathbb{R}^L | \forall_p \, p \cdot y \leq \pi(p)\right\}$. π is lower semi-continuous if for $p_n \to p$ it holds that $\pi(p) \leq \lim \pi(p_n)$.

Definition (Weak Axiom of Profit Maximization) Consider a set of observations $(y_t, p_t)_{t=1}^T$, it satisfies the WAPM if $p_t \cdot y_s \leq p_t \cdot y_t$ for all t, s. (at prices t it must be that y_t is profit maximizing, other production can't give higher profits).

Proposition (Rationalizability of Profits and WAPM) Observations $(y_t, p_t)_{t=1}^T$ satisfy WAPM if and only if there exists a closed, convex production set Y that profit-rationalizes these observations.

Proposition (Law of Supply) If observations $(y_t, p_t)_{t=1}^T$ satisfy WAPM they also satisfy the law of supply:

$$[p_t - p_s] \cdot [y_t - y_s] \geq 0$$

(if the price of a good increases its supply increases as well).

Note: Add $p_t \cdot y_s \leq p_t \cdot y_t$ and $p_s \cdot y_t \leq p_s \cdot y_s$ to obtain $(p_t - p_s) \cdot y_s \leq (p_t - p_s) \cdot y_t$, the result follows.

Definition (Weak Axiom of Cost Minimization) Let $(x_t, w_t, z_t)_{t=1}^T$ be observations on input choices, input prices and output quantities. It must be that:

$$\forall_{t,s \in \{1,\ldots,T\}} z_s \geq z_t \to w_t \cdot x_s \geq w_t \cdot x_t$$

Definition (Rationalizability of Production Function) Observations $(x_t, w_t, z_t)_{t=1}^T$ are rationalized by production function $f : \mathbb{R}_+^n \to \mathbb{R}_+$ if for all $t \in \{1, \ldots, T\}$ $x_t \in \operatorname{argmin}\{w_t \cdot x \text{ s.t. } f(x) \geq z_t\}$.

Proposition (Rationalizability and WACM) If a production function rationalizes observations $(x_t, w_t, z_t)_{t=1}^T$ then the observations satisfy the WACM.

7 Rationalizability in Consumer Choice

Definition (Rationalization - by a utility function) Consider a set of observations $(x_t, p_t)_{t=1}^T$, utility function $u : \mathbb{R}_+^n \to \mathbb{R}$ rationalizes the observations if for every t and $x \in \mathbb{R}_+^n$:

$$p_t \cdot x \le p_t \cdot x_t \to u(x) \le u(x_t)$$

If there exists u that rationalizes the observations and that is lns it must be that:

i. Income of the agent at t is $p_t \cdot x_t$.

ii. $p_t \cdot x \le p_t \cdot x_t \to u(x) \le u(x_t)$

iii. $p_t \cdot x < p_t \cdot x_t \to u(x) < u(x_t)$

Definition (Revealed Preferences) Consider an observation (x, p):

Weakly revealed preferred (xRy) x is weakly revealed preferred to y if $p \cdot y \le p \cdot x$.

Strictly revealed preferred (xPy) x is strictly revealed preferred to y if $p \cdot y < p \cdot x$.

Definition (Generalized Weak Axiom of Revealed Preferences) Consider a set of observations $(x_t, p_t)_{t=1}^T$, if $p_t \cdot x_s \le p_t \cdot x_t$ then $p_s \cdot x_t \ge p_s \cdot x_s$. (if x_t is revealed preferred to x_s -since x_s was affordable at price p_t- then x_t cannot be strictly affordable under p_s -or else there would be an affordable x' preferred to x_s-).

GWARP If $x_t R x_s$ then $\neg x_s P x_t$.

WARP If $x_t R x_s$ and $x_t \neq x_s$ then $\neg x_s R x_t$.

Definition (Generalized Axiom of Revealed Preferences) Consider a set of observations $(x_t, p_t)_{t=1}^T$, for any subset of observations $(x_{t_i}, p_{t_i})_{i \in I}$ if

$$x_{t_1} R x_{t_2} \wedge \ldots \wedge x_{t_{n-1}} R x_{t_n} \to \neg x_{t_n} P x_{t_1}$$

(If an observation is (indirectly) revealed weakly preferred to another then the second cannot be strictly preferred to the first).

Axiom of Revealed Preferences If for any subset of observations

$$x_{t_1} R x_{t_2} \wedge \ldots \wedge x_{t_{n-1}} R x_{t_n} \wedge x_{t_1} \neq x_{t_n} \to \neg x_{t_n} R x_{t_1}$$

Proposition (Necessity of GWARP) If u rationalizes $(x_t, p_t)_{t=1}^T$ and is lns then $(x_t, p_t)_{t=1}^T$ satisfy GWARP.

Theorem (Afriat) Observations $(x_t, p_t)_{t=1}^T$ satisfy GARP if and only if there exists a lns utility functionu that rationalizes these observations.

8 Monotone Comparative Statics

Definition (Lattice operators) Let $x, y \in \mathbb{R}^n$. Lattice operators $\wedge$ and $\vee$ are defined as:

$$x \wedge y = \begin{bmatrix} \min[x_1, y_1] \\ \vdots \\ \min[x_n, y_n] \end{bmatrix} \quad \text{and} \quad x \vee y = \begin{bmatrix} \max[x_1, y_1] \\ \vdots \\ \max[x_n, y_n] \end{bmatrix}$$

Definition (Lattice) A set $X \subseteq \mathbb{R}^n$ is a lattice is $\forall_{x,y \in X} x \wedge y \in X \quad \text{and} \quad x \vee y \in X$.

Definition (Supermodular function) A function $f : X \to \mathbb{R}$ on a lattice X is supermodular on X if:

$$\forall_{x,y \in X} f(x \vee y) - f(x) \geq f(y) - f(x \wedge y)$$

Definition (Strong set order) Let $A, B \subseteq \mathbb{R}^n$.

$$A \leq_{sso} B \iff \forall_{x \in A} \forall_{y \in B} x \wedge y \in A \quad \text{and} \quad x \vee y \in B$$

Note: If A and B are singletons the strong set order is the usual inequality between vectors.

Definition (Monotone non-decreasing or non-increasing sets) Let $t \in \mathbb{R}^m$ and $\varphi(t) : \mathbb{R}^m \to \mathbb{R}^n$ a correspondence. φ is monotone non-decreasing in t if:

$$\forall_{t,t'} t \leq t' \longrightarrow \varphi(t) \leq_{sso} \varphi\left(t'\right)$$

it is monotone non-increasing if instead $\varphi\left(t'\right) \leq_{sso} \varphi(t)$.

Definition (Non-decreasing (non-increasing) differences) Let $X \subseteq \mathbb{R}^n$, $T \subseteq \mathbb{R}^m$ and $f : X \times T \to \mathbb{R}$. $f(x,t)$ has non-decreasing differences in (x,t) if for $x' \geq x$ the difference $f\left(x', t\right) - f(x,t)$ is non-decreasing in t. It has non-increasing differences if $f\left(x', t\right) - f(x,t)$ is non-increasing in t.

Note: These conditions are equivalent to:

$$\forall_{x \geq x'} \forall_{t' \geq t} f\left(x', t'\right) - f\left(x, t'\right) \geq (\leq) f\left(x', t\right) - f(x,t)$$

Proposition (Supermodularity in $\mathbb{R}$ and $\mathbb{R}^2$)

i. A function $f : \mathbb{R} \to \mathbb{R}$ is always supermodular.

ii. A function $f : \mathbb{R}^2 \to \mathbb{R}$ is supermodular if and only if it has non-decreasing differences.

Proposition (Supermodularity and differentiability) Let $f : \mathbb{R}^n_+ \to \mathbb{R}$ be twice continuously differentiable. f is supermodular in $\mathbb{R}^n_+$ if and only if

$$\forall_{x \in \mathbb{R}^n_+} \forall_{i \neq j} \frac{\partial^2 f}{\partial x_i \partial x_j}(x) \geq 0$$

Proposition (Non-decreasing differences and differentiability) Let $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ be twice differentiable on $X \subseteq \mathbb{R}^n \times \mathbb{R}^m$. f has non-decreasing (non-increasing) differences in (x, t) on the X if and only if

$$\forall_{(x,t)\in X} \forall_i \forall_j \frac{\partial^2 f}{\partial x_i \partial t_j}(x, t) \geq (\leq)\, 0$$

Proposition (Maximizer properties - Topkis) Let $S \subseteq X \subseteq \mathbb{R}^n$, $T \subseteq \mathbb{R}^m$ and $f : X \times T \to \mathbb{R}$. Consider the problem of maximizing f over S and its set of solutions φ:

$$\max_{x\in S} f(x, t) \qquad \varphi(t) = \{x \in S | \forall_y f(x, t) \geq f(y, t)\}$$

i. If f is supermodular in x and S is a lattice then $\varphi(t)$ is a lattice.

ii. (Topkis) $\varphi(t)$ is monotone non-decreasing (non-increasing) in t if:

  (a) S is a lattice.

  (b) f is supermodular in x.

  (c) f has non-decreasing (non-increasing) differences in (x, t).

Note: If in addition $\varphi(t)$ is a continuous correspondence then the supremum $(\bar{\varphi}(t))$ and infimum $(\underline{\varphi}(t))$ of the set belong to $\varphi(t)$ and are non-decreasing (non-increasing) functions.

9 Choice Under Uncertainty

Let there be S states of nature $s \in \{1, \ldots, S\}$. There is a single commodity. A state contingent consumption plan $c \in \mathbb{R}_+^S$ indicates the agent's consumption in each state of nature. Let $\succeq$ be a preference relation on the set $\mathbb{R}_+^S$ of consumption plans. $\succeq$ is assumed rational, strictly increasing and continuous.

Definition (State-separable utility representation) $\succeq$ has a state-separable utility representation if there exist functions $v_s : \mathbb{R}_+ \to \mathbb{R}$ for $s \in \{1, \ldots, S\}$ such that:

$$\forall_{c,c' \in \mathbb{R}_+^S} c \succeq c^{'} \iff \sum_{s=1}^{S} v_s(c_s) \geq \sum_{s=1}^{S} v_s\left(c_s^{'}\right)$$

Definition (VNM - Expected utility representation (with respect to π)) $\succeq$ has an expected utility representation with respect to $\pi \in \Delta_S$ if there exist a function $v : \mathbb{R}_+ \to \mathbb{R}$ such that:

$$\forall_{c,c' \in \mathbb{R}_+^S} c \succeq c^{'} \iff E_\pi[v(c)] = \sum_{s=1}^{S} \pi_s v(c_s) \geq \sum_{s=1}^{S} \pi_s v\left(c_s^{'}\right) = E_\pi\left[v\left(c^{'}\right)\right]$$

Axiom (Independence - Sure thing) Let $c \in \mathbb{R}_+^S$ and $y \in \mathbb{R}_+$. The consumption plan $c_{-s}y = [c_1, \ldots, c_{s-1}, y, c_{s+1}, \ldots, c_S]^{'}$ is defined by replacing consumption contingent on state s with y.

$$\forall_{c,d \in \mathbb{R}_+^S} \forall_{y,w \in \mathbb{R}_+} \forall_s \quad c_{-s}y \succeq d_{-s}y \iff c_{-s}w \succeq d_{-s}w$$

Definition (Risk Aversion)

i. Preference relation $\succeq$ is risk averse (with respect to $\pi \in \Delta_S$) if:

$$\forall_c \quad E_\pi[c] \succeq c$$

If $\succeq$ has an expected utility (VNM) representation this condition is:

$$\forall_c \quad v(E_\pi[c]) \geq E_\pi[v(c)]$$

ii. Preference relation $\succeq$ is risk neutral (with respect to $\pi \in \Delta_S$) if:

$$\forall_c \quad E_\pi[c] \sim c$$

If $\succeq$ has an expected utility (VNM) representation this condition is:

$$\forall_c \quad v(E_\pi[c]) = E_\pi[v(c)]$$

Definition (Risk Compensation) Let $x \in \mathbb{R}_+$ and $\tilde{z} \in \mathbb{R}^S$ such that for $\pi \in \Delta_S$ $E_\pi[\tilde{z}] = 0$. Risk compensation with respect to $\tilde{z}$ is $\rho(x,z) \in \mathbb{R}$ such that:

$$x + \tilde{z} \sim x - \rho(x, \tilde{z})$$

If $\succeq$ has an expected utility (VNM) representation this condition is:

$$E_\pi[v(x+\tilde{z})] = v(x - \rho(x,\tilde{z}))$$

Definition (Arrow-Pratt measures of risk aversion) Let $\succeq$ be a strictly increasing preference relation with expected utility representation v. If v is twice differentiable, the Arrow-Pratt absolute and relative risk aversion are defined as:

$$A(x) = -\frac{v''(x)}{v'(x)} \qquad \text{and} \qquad R(x) = -\frac{v''(x)}{v'(x)}x$$

Proposition (State-separable representation - Debreu (1959)) Let $S \geq 3$ and $\succeq$ a rational, strictly increasing and continuous preference relation. Then $\succeq$ has a state-separable utility representation if and only if it obeys the independence axiom.

Note: With $S = 2$ the independence axiom is satisfies immediately by all strictly increasing preferences.

Proposition (Expected utility representation) Let $S \geq 3$ and $\succeq$ a rational, strictly increasing and continuous preference relation. Then $\succeq$ satisfies the independence axiom and is risk averse, with respect to $\pi \in \Delta_S$, if and only if it has a **concave** expected utility representation with respect to $\pi \in \Delta_S$.

Proposition (State separable and expected utility representation) Let preferences be represented by $U = \sum_{s=1}^{S} u_s(c_s)$ if the agent is risk averse with respect to probability measure π, then utility function U must have an expected utility representation under π. Moreover the expected utility is concave.

Proof (Differentiable case)

- An agent is risk averse if $U(c) \leq U(E[c])\ \forall c$.
- Consider the following problem:

$$\max_{c \in \mathbb{R}^S_+} U(c) = \sum_{s=1}^{S} u_s(c_s) \qquad \text{s.t. } E[c] = \bar{c}$$

- By risk aversion it must be that $c = (\bar{c}, \ldots, \bar{c})$ is a solution to the problem, then it has to satisfy the FOC for all s:

$$u_s'(\bar{c}) = \pi_s \lambda$$

- Taking the ration of any two: $u_s^{'}(\overline{c}) = \frac{\pi_s}{\pi_{\hat{s}}} u_{\hat{s}}^{'}(\overline{c})$ which implies for all s: $u_s(x) = \frac{\pi_s}{\pi_{\hat{s}}} u_{\hat{s}}(x) + k_s$ (by integration).
- Defining $v(x) = u_{\hat{s}}(x)$ one can write the utility function as:

$$U(c) = \sum_{s=1}^{S} u_s(c_s) = \sum_{s=1}^{S} \frac{\pi_s}{\pi_{\hat{s}}} v(c_s) + \sum_{s=1}^{S} k_s$$

- Note that if U represents some preference relation $\succeq$ then $aU+b$ for $a \in \mathbb{R}_{++}$ and $b \in \mathbb{R}$ also represents the same preferences $\left(U(c) \geq U\left(c^{'}\right) \iff aU(c) + b \geq aU\left(c^{'}\right) + b\right)$, then the function V:

$$V(c) = \pi_{\hat{s}} U(c) - \pi_{\hat{s}} \sum_{s=1}^{S} k_s = \sum_{s=1}^{S} \pi_s v(c_s) = E_\pi[v]$$

  represents the same preferences than U. V is of the expected utility form. Note that v is increasing.

Proposition (Risk aversion and risk compensation) Let $\succeq$ be a rational, strictly increasing and continuous preference relation. Then

i. $\succeq$ is risk averse if and only if $\forall_{x \in \mathbb{R}_+} \forall_{\tilde{z} \in \mathbb{R}^S, E[\tilde{z}]=0} \quad \rho(x, \tilde{z}) \geq 0$.

ii. $\succeq$ is risk neutral if and only if $\forall_{x \in \mathbb{R}_+} \forall_{\tilde{z} \in \mathbb{R}^S, E[\tilde{z}]=0} \quad \rho(x, \tilde{z}) = 0$.

Proposition (Theorem of Pratt) Let $v_1, v_2 \in C^2$ strictly increasing VNM utility functions (with respect to $\pi \in \Delta_S$), with risk compensations ρ_1 and ρ_2, and Arrow-Pratt absolute risk aversion measures A_1 and A_2 respectively. The following conditions are equivalent:

i. $\forall_{x \in \mathbb{R}} A_1(x) \geq A_2(x)$

ii. $\forall_{x \in \mathbb{R}} \forall_{\tilde{z} \in \mathbb{R}^S, E[\tilde{z}]=0} \rho_1(x, \tilde{z}) \geq \rho_2(x, \tilde{z})$

iii. v_1 is a concave transformation of v_2. For some f be concave and strictly increasing, $\forall_{x \in \mathbb{R}} v_1(x) = f(v_2(x))$

Note: If $\forall_{x \in \mathbb{R}} \forall_{\tilde{z} \in \mathbb{R}^S, E[\tilde{z}]=0} \rho_1(x, \tilde{z}) \geq \rho_2(x, \tilde{z})$ we say that v_1 is more risk averse than v_2. From the theorem we know that this is equivalent to $\forall_{x \in \mathbb{R}} A_1(x) \geq A_2(x)$.

Corollary 1 (Theorem of Pratt)

i. A consumer is risk averse if and only if his VNM utility function is concave.

ii. A consumer is risk neutral if and only if his VNM utility function is linear.

iii. A consumer is strictly risk averse if and only if his VNM utility function is strictly concave.

Note: This also works without the differentiability assumption.

Corollary 2 (Theorem of Pratt)

i. $\rho(x, \tilde{z})$ is increasing in x for every $\tilde{z}$ with $E_\pi[\tilde{z}] = 0$ if and only if $A(x)$ is increasing in x.

ii. $\rho(x, \tilde{z})$ is constant in x for every $\tilde{z}$ with $E_\pi[\tilde{z}] = 0$ if and only if $A(x)$ is constant in x.

iii. $\rho(x, \tilde{z})$ is decreasing in x for every $\tilde{z}$ with $E_\pi[\tilde{z}] = 0$ if and only if $A(x)$ is decreasing in x.

Note: For the proof let v be a VNM utility function and and define $v_1(y) \equiv v(y + \Delta y)$ for $\Delta y \geq 0$. Note that $A_1(y) = A(y + \Delta y)$ and $\rho_1(y, \tilde{z}) = \rho(y + \Delta y, \tilde{z})$.

9.1 Ellsberg Paradox

Consider an urn with a fixed number of balls. There are three possible colors, say red, blue and green. The proportion of red balls is known and is noted as $\pi_r < 1/2$.

Now consider bets of \$1 on a ball of a certain color (or colors) being drawn from the urn. The payoff is \$1 if right and \$0 if wrong.

The paradox consists in the agent having a VNM utility and observed preferences over bets being: $1_r \succ 1_b$ and $1_{b\vee g} \succ 1_{r\vee g}$. This implies for the utility that (denoting $v(1) = v_1$ and $v(0) = v_0$:

$$\begin{aligned} E\left[v\left(1_r\right)\right] &> E\left[v\left(1_b\right)\right] & \qquad E\left[v\left(1_{b\vee g}\right)\right] &> E\left[v\left(1_{r\vee g}\right)\right] \\ \pi_r v_1 + \left(\pi_b + \pi_g\right) v_0 &> \pi_b v\left(1\right) + \left(\pi_r + \pi_g\right) v_0 & \qquad \pi_r v_0 + \left(\pi_b + \pi_g\right) v_1 &> \pi_b v_0 + \left(\pi_r + \pi_g\right) v_1 \\ \pi_r\left(v_1 - v_0\right) &> \pi_b\left(v_1 - v_0\right) & \qquad \pi_b\left(v_1 - v_0\right) &> \pi_r\left(v_1 - v_0\right) \end{aligned}$$

Since $v(\cdot)$ is increasing $v_1 - v_0 \geq 0$, the inequalities would imply $\pi_r > \pi_b$ and $\pi_b > \pi_r$, which is a contradiction.

Multiple prior expected utility If the agent has a different utility function the paradox can be avoided. Let $\mathcal{P}$ be the set of all possible probability distributions over the distributions of the balls subject to the known information ($\Pr\{r\} = \overline{\pi} < 1/2$ and $\Pr\{b\} + \Pr\{g\} = 1 - \overline{\pi}$).

The agent's utility is defined as: $u(B) = \min_{p\in\mathcal{P}} E_p\left[v(B)\right]$, where $v(B)$ is the VNM utility function evaluated at the bet's payoffs.

In this case preference $1_r \succ 1_b$ implies:

$$\begin{aligned} \min_{p\in\mathcal{P}}\left\{\pi_r(p) v_1 + \left(\pi_b(p) + \pi_g(p)\right) v_0\right\} &> \min_{p\in\mathcal{P}}\left\{\pi_b(p) v_1 + \left(\pi_r(p) + \pi_g(p)\right) v_0\right\} \\ \min_{p\in\mathcal{P}}\left\{\overline{\pi} v_1 + \left(1 - \overline{\pi}\right) v_0\right\} &> \min_{p\in\mathcal{P}}\left\{\pi_b(p) v_1 + \left(\overline{\pi} + \pi_g(p)\right) v_0\right\} \\ \overline{\pi} v_1 + \left(1 - \overline{\pi}\right) v_0 &> 0 v_1 + \left(\overline{\pi} + \left(1 - \overline{\pi}\right)\right) v_0 \\ \overline{\pi}\left(v_1 - v_0\right) &> 0 \end{aligned}$$

Since there is no uncertainty over the expected payoff of the bet 1_r and the expected minimum payoff of the bet 1_b occurs when the probability of a blue ball is zero. The last inequality holds since $\overline{\pi} > 0$ and $v(\cdot)$ is increasing.

In this case preference $1_{b\vee g} \succ 1_{r\vee g}$ implies:

$$\begin{aligned} \min_{p\in\mathcal{P}}\left\{\pi_r(p) v_0 + \left(\pi_b(p) + \pi_g(p)\right) v_1\right\} &> \min_{p\in\mathcal{P}}\left\{\pi_b(p) v_0 + \left(\pi_r(p) + \pi_g(p)\right) v_1\right\} \\ \min_{p\in\mathcal{P}}\left\{\overline{\pi} v_0 + \left(1 - \overline{\pi}\right) v_1\right\} &> \min_{p\in\mathcal{P}}\left\{\pi_b(p) v_0 + \left(\overline{\pi} + \pi_g(p)\right) v_1\right\} \\ \overline{\pi} v_0 + \left(1 - \overline{\pi}\right) v_1 &> \left(1 - \overline{\pi}\right) v_0 + \left(\overline{\pi} + 0\right) v_1 \\ \left(1 - \overline{\pi}\right)\left(v_1 - v_0\right) &> \overline{\pi}\left(v_1 - v_0\right) \end{aligned}$$

Since there is no uncertainty over the expected payoff of the bet $1_{b\vee g}$ and the expected minimum payoff of the bet $1_{r\vee g}$ occurs when the probability of a blue ball is $1 - \overline{\pi}$. The last inequality holds since $\overline{\pi} < 1/2$ and $v(\cdot)$ is increasing.

9.2 Stochastic Dominance and Risk

Let $\tilde{z}$ and $\tilde{y}$ be random variables that take values on the interval $[a, b]$. Let $F_{\tilde{z}}(x) = \Pr\{\tilde{z} \leq x\}$ the Cumulative Distribution Function (CDF).

Definition (First order stochastic dominance) $\tilde{z}$ first order stochastically dominates (FSD) $\tilde{y}$ if:

$$\forall_{t\in[a,b]} F_{\tilde{z}}(t) \leq F_{\tilde{y}}(t)$$

Definition (Second order stochastic dominance) $\tilde{z}$ second order stochastically dominates (SSD) $\tilde{y}$ if:

$$\forall_{t\in[a,b]} \int_a^t F_{\tilde{z}}(x)\,dx \leq \int_a^t F_{\tilde{y}}(x)\,dx$$

Note: Since $E[\tilde{z}] = b - \int_a^b F_{\tilde{z}}(x)\,dx$ this implies $E[\tilde{z}] \geq E[\tilde{y}]$.
Note: If $\tilde{z}$ FSD $\tilde{y}$ then $\tilde{z}$ SSD $\tilde{y}$.

Definition (Riskiness) $\tilde{y}$ is more risky than $\tilde{z}$ if $\tilde{z}$ SSD $\tilde{y}$ and $E[\tilde{z}] = E[\tilde{y}]$.

Proposition (FSD and non-decreasing continuous functions) $\tilde{z}$ first order stochastically dominates (FSD) $\tilde{y}$ if and only if for every function $v : \mathbb{R} \to \mathbb{R}$ such that v is non-decreasing and continuous:

$$E[v(\tilde{z})] \geq E[v(\tilde{y})]$$

Proposition (SSD and concave non-decreasing continuous functions) $\tilde{z}$ second order stochastically dominates (SSD) $\tilde{y}$ if and only if for every function $v : \mathbb{R} \to \mathbb{R}$ such that v is non-decreasing, continuous and concave:

$$E[v(\tilde{z})] \geq E[v(\tilde{y})]$$

Proposition (Scalar product and riskiness) Let $\tilde{z}$ be such that $E[\tilde{z}] = 0$ and $k_1 \geq k_2$ then $k_1\tilde{z}$ is more risky that $k_2\tilde{z}$.

Note: Use $E[v(\tilde{z})] \geq E[v(\tilde{y})]$ with concave v and note $k_2\tilde{z} = \frac{k_2}{k_1}(k_1\tilde{z}) + \left(1 - \frac{k_2}{k_1}\right) E[k_1\tilde{z}]$.

Proposition (Riskiness and variance) If $\tilde{y}$ is more risky than $\tilde{z}$ then $V[\tilde{y}] \geq V[\tilde{z}]$ where $V[\tilde{z}] = E\left[(\tilde{z} - E[\tilde{z}])^2\right]$.

Note: The converse is not true when $\tilde{y}$ and $\tilde{z}$ can take more than two values.
Note: Use $v(x) = -(\alpha - x)^2$ with $\alpha \geq x$ and $E[v(\tilde{z})] \geq E[v(\tilde{y})]$ to prove $E[\tilde{y}^2] \geq E[\tilde{z}^2]$. Note they have the same mean.

Proposition (Riskiness and increasing affine transformations)

$$\tilde{z}\text{SSD}\tilde{y} \to \forall_{a\geq 0}\forall_{b\in\mathbb{R}} (a\tilde{y} + b)\,\text{SSD}\,(a\tilde{z} + b)$$

Proposition (Riskiness with several variables) Let $\tilde{z}$ and $\tilde{y}$ be nondeterministic random variables such that $\tilde{y}$ has mean zero, and is mean independent of $\tilde{z}$, that is, $E[\tilde{y}|z] = E[\tilde{y}] = 0$. Then $\tilde{z} + \tilde{y}$ is more risky than $\tilde{z}$, $\tilde{z}\mathrm{SSD}(\tilde{z} + \tilde{y})$.

Proof: Let v be a concave and non-decreasing function, then:

$$\begin{aligned} E[v(\tilde{z} + \tilde{y})] = E_z[E_y[v(\tilde{z} + \tilde{y})|z]] &\leq E_z[v(E_y[\tilde{z} + \tilde{y}|z])] = E_z[v(\tilde{z} + E_y[\tilde{y}|z])] \\ &= E_z[v(\tilde{z} + E[\tilde{y}])] = E[v(\tilde{z})] \end{aligned}$$

where the inequality follows from Jensen's inequality $(E_y[v(\tilde{z} + \tilde{y})|z] \leq v(E_y[\tilde{z} + \tilde{y}|z]))$ and the first step form the law of iterated expectations.

Note: The requirement of mean independence is stronger than that of no correlation between $\tilde{z}$ and $\tilde{y}$, but weaker than that of independence.

Part III
Beth Allen

10 Preferences

In what follows $\succeq$ is a (preference) relation on a (consumption) set X. It is assumed to be a complete preorder (complete, transitive and reflexive). The definitions (including continuity) follow Debreu (1987).

Continuity

Sequential Definition For any sequence $\{(x_n, y_n)\}$ with $\forall_n x_n \succeq y_n$, $x_n \to x$ and $y_n \to y$ we have $x \succeq y$.

Set Definition For all $x \in X$ the upper contour $(U(x) = \{y \in X | y \succeq x\})$ and lower contour $(L(x) = \{y \in X | x \succeq y\})$ are closed.

Monotonicity

Definition (Weakly monotone) A preorder $\succeq$ is weakly monotone on a set X if:

$$\forall_{x,y \in X} x \geq y \to x \succeq y$$

Definition (Monotone) A preorder $\succeq$ is monotone on a set X if:

$$\forall_{x,y \in X} x \gg y \to x \succ y$$

Definition (Strongly monotone) A preorder $\succeq$ is strongly monotone on a set X if:

$$\forall_{x,y \in X} x \geq y \wedge x \neq y \to x \succ y$$

Convexity

Definition (Weakly convex) A preorder $\succeq$ is weakly convex on a set X if:

$$\forall_{x,y \in X} \forall_{\lambda \in (0,1)} x \succeq y \to \lambda x + (1 - \lambda) y \succeq y$$

Definition (Convex) A preorder $\succeq$ is convex on a set X if:

$$\forall_{x,y \in X} \forall_{\lambda \in (0,1)} x \succ y \to \lambda x + (1 - \lambda) y \succ y$$

Definition (Strongly -Strictly- convex) A preference relation $\succeq$ is strongly convex on a set X if:

$$\forall_{x,y \in X} \forall_{\lambda \in (0,1)} x \sim y \wedge x \neq y \to \lambda x + (1 - \lambda) y \succ y$$

Proposition If $\succeq$ is a convex and continuous, then it is weakly convex.

Proposition If $\succeq$ is a strongly convex and continuous, then it is convex.

Proposition If $\succeq$ is a weakly convex, continuous and locally non-satiated, then it is convex.

Proof:

- Suppose for contradiction that $\succeq$ is not convex. Then there exists $x, y \in \mathbb{R}_{+}^{l}$ and $\lambda^{'} \in (0,1)$ such that $x \succ y$ and $y \succeq \lambda^{'} x + \left(1 - \lambda^{'}\right) y = z$.
- By weak convexity $z \succeq y$, then $z \sim y$.
- Let $\epsilon_n = \frac{1}{n}$ by l.n.s. there exists $y_n \in B_{\epsilon_n}(y)$ such that $y_n \succ y$. Let

$$A_n = \left\{ q \in \mathbb{R}_{+}^{l} | \exists_{\alpha > 0} q = \alpha \left(z - y_n\right) + y_n \right\}$$

  and define $x_n = \underset{q \in A_n}{\operatorname{argmin}} \|x - q\|$. Note that by construction $y_n \to y$ and then $x_n \to x$, also that for all n, there exists $\alpha_n \in (0,1)$ such that $z = \alpha_n x_n + (1 - \alpha_n) y_n$.
- It is the case that there exists N such that $x_n \succ y$. Suppose for a contradiction that for all n $y \succeq x_n$. Then by continuity $y \succeq \lim x_n = x$ which contradicts $x \succ y$.
- If $x_N \succeq y_N$ we have by weak convexity $z \succeq y_N \succ y$ which is a contradiction with $z \sim y$.
- If $y_N \succeq x_N$ we have by weak convexity $z \succeq x_N \succ y$ which is a contradiction with $z \sim y$.
- Then $\succeq$ is convex.

Proposition $\succeq$ is weakly convex if and only if the (open) upper contours are convex ($\{y \in X | y \succ x\}$ is convex).

Proposition $\succeq$ is convex if and only if the upper contours are convex ($\{y \in X | y \succeq x\}$ is convex).

Corollary If $\succeq$ is convex it can only be represented by a quasi-concave utility function.

11 Auxiliary Theorems and Definitions

Definition (Upper semi-continuous function - u.s.c.) A function $f : \mathbb{R}^n \to \mathbb{R}$ is u.s.c. if $\forall_{y\in\mathbb{R}} \{x \in \mathbb{R}^n | f(x) < y\}$ is open in $\mathbb{R}^n$.

Definition (Lower semi-continuous function - l.s.c.) A function $f : \mathbb{R}^n \to \mathbb{R}$ is l.s.c. if $\forall_{y\in\mathbb{R}} \{x \in \mathbb{R}^n | f(x) > y\}$ is open in $\mathbb{R}^n$.

Definition (Upper hemi-continuous correspondence - u.h.c.) A *compact valued* correspondence $\Gamma : X \rightrightarrows Y$ is u.h.c. at $x \in X$ if for every $\{x_n\} \subset X$ such that $x_n \to x$ and every $\{y_n\} \subset Y$ such that $y_n \in \Gamma(x_n)$ there exits a convergent subsequence $\{y_{n_k}\}$ such that $y_{n_k} \to y \in \Gamma(x)$.

$$\forall_{x_n \to x} \forall_{y_n \in \Gamma(x_n)} \exists_{\{y_{n_k}\}} y_{n_k} \to y \in \Gamma(x)$$

Definition (Lower hemi-continuous correspondence - l.h.c.) A correspondence $\Gamma : X \rightrightarrows Y$ is l.h.c. at $x \in X$ if for all $y \in \Gamma(x)$ and all sequences $\{x_n\} \subset X$ such that $x_n \to x$ there exits a sequence $\{y_n\} \subset Y$ such that $y_n \in \Gamma(x_n)$ and $y_n \to y$.

$$\forall_{x_n \to x} \forall_{y \in \Gamma(x)} \exists_{y_n \in \Gamma(x_n)} y_n \to y$$

Definition (Closed correspondence or Closed graph property) A correspondence $\Gamma : X \rightrightarrows Y$ is closed if $\mathrm{Gr}(\Gamma) = \{(x, y) | x \in X \wedge y \in \Gamma(x)\}$ is a closed subset of $X \times Y$. This is, if for every $x \in X$ and $\{x_n\} \subset X$ such that $x_n \to x$ and every $\{y_n\} \subset Y$ such that $y_n \in \Gamma(x_n)$ and $y_n \to y$ we have $y \in \Gamma(x)$.

$$\forall_x \forall_{x_n \to x} \forall_{y_n \in \Gamma(x_n)} y_n \to y \implies y \in \Gamma(x)$$

Note: If Γ has a closed graph then it is closed valued. Moreover, if Y is compact, Γ is compact valued. The converse is not true.

Proposition (u.h.c and Closed graph) Let $\Gamma : X \rightrightarrows Y$. If Γ is u.h.c, then Γ is closed (has a closed graph).

Proof: Let Γ be u.h.c. Take $x \in X$, $x_n \to x$ and $\{y_n\} \subset Y$ such that $y_n \in \Gamma(x_n)$ and $y_n \to y$. Since Γ is u.h.c. there is a convergent subsequence $\{y_{n_k}\}$ of $\{y_n\}$ such that $y_{n_k} \to y' \in \Gamma(x)$. Since $y_n \to y$ it follows that $y = y'$ and then $y \in \Gamma(x)$. Then Γ is closed.

Proposition (Closed graph and u.h.c.) Let $\Gamma : X \rightrightarrows Y$. If Y is compact and Γ is closed (has a closed graph), then Γ is u.h.c.

Proof: Let Y be compact and Γ closed. First note that this implies that Γ is compact valued (since closed graph implies closed valued). Take $x \in X$, $x_n \to x$ and $\{y_n\} \subset Y$ such that $y_n \in \Gamma(x_n)$. Since Y is compact $\{y_n\}$ has a convergent subsequence $y_{n_k} \to y$. Since Γ is closed it follows that $y \in \Gamma(x)$. Then Γ is u.h.c.

Proposition (Cartesian product of closed correspondences) Let $\Gamma_1 : X_1 \rightrightarrows Y_1$ and $\Gamma_2 : X_2 \rightrightarrows Y_2$ be closed and define $\Gamma : X_1 \times X_2 \rightrightarrows Y_1 \times Y_1$ as $\Gamma(x_1, x_2) = (\Gamma_1(x_1), \Gamma(x_2))$. Then Γ is closed.

Proof: Let $(x_1, x_2) \in X_1 \times X_2$, $(x_{1n}, x_{2n}) \to (x_1, x_2)$ and $\{(y_{1n}, y_{2n})\} \subset Y_1 \times Y_2$ such that $(y_{1n}, y_{2n}) \in \Gamma(x_{1n}, x_{2n})$ and $(y_{1n}, y_{2n}) \to (y_1, y_2)$. Then $y_{1n} \in \Gamma_1(x_{1n})$, $y_{2n} \in \Gamma_2(x_{2n})$ and $y_{1n} \to y_1$, $y_{2n} \to y_2$. Since Γ_1 and Γ_2 are closed we know that $y_1 \in \Gamma_1(x_1)$ and $y_2 \in \Gamma_2(x_2)$. This implies $(y_1, y_2) \in \Gamma(x_1, x_2)$, hence Γ is closed.

Theorem (Maximum -ToM-) Let $X \subset \mathbb{R}^n$ and $Y \subset \mathbb{R}^m$, let $f : X \times Y \to \mathbb{R}$ be a continuous function and $\Gamma : X \rightrightarrows Y$ a nonempty, compact valued, continuous correspondence. Define:

$$v(x) = \max_{y \in \Gamma(x)} f(x, y) \qquad\qquad G(x) = \{y \in \Gamma(x) \,|\, f(x, y) = v(x)\}$$

Then $v : X \to Y$ is continuous, and $G : X \rightrightarrows Y$ is nonempty and compact valued, and u.h.c.

Theorem (Maximum under convexity - Sundaram (1996))

i. If $f(x, \cdot)$ is concave for all x and Γ is convex valued then G is convex valued. Moreover, if $f(x, \cdot)$ is strictly concave then G is single valued, hence a continuous function.

ii. If f is concave on $X \times Y$ and Γ has a convex graph then v is concave and G is convex valued. Moreover, if f is strictly concave on $X \times Y$ then v is strictly concave and G is single valued, hence a continuous function.

Corollary (Quasi-Concavity) If $f(x, \cdot)$ is quasi-concave for all x and Γ is convex valued then G is convex valued. Moreover, if $f(x, \cdot)$ is strictly quasi-concave then G is single valued, hence a continuous function.

Theorem (Brouwer) Let $S \subset \mathbb{R}^n$ be nonempty, compact and convex, and $f : S \to S$ be a continuous function. Then f has a fixed point in S ($\exists_{\overline{x} \in S} f(\overline{x}) = \overline{x}$).

Theorem (Kakutani) Let $S \subset \mathbb{R}^n$ be nonempty, compact and convex, and $\Gamma : S \rightrightarrows S$ be a nonempty valued, convex valued and u.h.c. correspondence. Then Γ has a fixed point in S ($\exists_{\overline{x} \in S} \overline{x} \in \Gamma(\overline{x})$).

Note: Since S is compact u.h.c is equivalent to Γ having a closed graph.

12 Existence

Definition (Exchange Economy) An exchange economy (with n traders and l goods) is formed by tuples of preferences, endowments and consumption sets that characterize traders (indexed by i). An economy is $\mathcal{E} = \{(\succeq_i, e_i, X_i)_{i=1}^n\}$, where $\succeq_i$ is a complete preorder that represents trader i's preferences on X_i, and $e_i \in X_i$ is the vector of initial endowments of that trader, consumption sets are $X_i \subseteq \mathbb{R}_+^l$.

Note: In most cases $X_i = \mathbb{R}_+^l$ and is therefore omitted from the definition of the economy. Alternatively the economy can be defined with u_i, a utility function that represents trader i's preferences. It is also assumed that $\sum_{i=1}^n e_i \gg 0$, this is, that there are not irrelevant goods.

Definition (Budget correspondence) The budget correspondence $B_i : \Delta \rightrightarrows \mathbb{R}_+^l$ is:

$$B_i(p, e_i) = \{x \in X_i | p \cdot x \leq p \cdot e_i\}$$

Definition (Demand correspondence) The demand correspondence of trader i is:

$$x_i(p, e_i) = \left\{x \in B_i(p, e_i) \,|\forall_{x' \in B_i(p,e_i)} x \succeq_i x^{'}\right\}$$

Aggregate excess demand correspondence:

$$Z(p) = \sum_{i=1}^n x_i(p, e_i) - \sum_{i=1}^n e_i$$

Definition (Competitive Equilibrium) CE $= \{p^\star, (x_i^\star)_{i=1}^n\} \in \overline{\Delta} \times \mathbb{R}_+^{ln}$ is a competitive equilibrium in $\mathcal{E}$ if $\forall_i x_i^\star \in x_i(p^\star, e_i)$ and $\sum_{i=1}^n x_i^\star = \sum_{i=1}^n e_i$. That is, $x_i^\star$ is in the demand correspondence of each agent, given p, $\succeq$ and e_i, and all markets clear.

Note: This conditions can be summarized as $0 \in Z(p^\star)$. Where $Z : \Delta \rightrightarrows \mathbb{R}^l$ is the aggregate excess correspondence. When $Z(\cdot)$ is a function we have $Z(p^\star) = 0$.

Definition (Boundary condition)

Function Let $Z : \Delta \to \mathbb{R}^l$ be an excess demand function. Z satisfies the boundary condition if $\forall_{\{p_n\} \subset \Delta} p_n \to p \in \partial\Delta \implies \|Z(p_n)\| \to \infty$.

Correspondence Let $\psi : \Delta \to \mathbb{R}^l$ be an excess demand correspondence. ψ satisfies the boundary condition if $\forall_{\{p_n\} \subset \Delta} p_n \to p \in \partial\Delta \implies \left[\forall_{\{Z_n\}} Z_n \in \psi(p_n) \implies \|Z_n\| \to \infty\right]$.

Note: A sufficient condition for the boundary condition to hold is strict monotonicity of preferences. Monotonicity is not sufficient (take Leontief preferences).

Theorem (Very easy existence theorem) Let $Z : \overline{\Delta} \to \mathbb{R}^l$ be an excess demand function. If Z is continuous and satisfies Walras' law $\left(\forall_{p\in\overline{\Delta}} p \cdot Z(p) = 0\right)$, then there exists $p^\star \in \overline{\Delta}$ such that $Z(p^\star) \leq 0$. Moreover $Z(p^\star) = 0$ if $p^\star \in \Delta$.

Outline:

- Note that $\overline{\Delta}$ is nonempty, compact and convex.
- Define $F : \overline{\Delta} \to \overline{\Delta}$ such that F is continuous. By Brouwer F has a fixed point on $\overline{\Delta}$.
- Argue that the fixed point is a competitive equilibrium.

Proof:

- Let $F(p) = \frac{1}{1+\sum \max(0,Z_i(p))} \left(p_1 + \max(0, Z_i(p)), \ldots, p_l + \max(0, Z_l(p))\right)$.
- Since $Z(\cdot)$ is continuous, then $Z_i(\cdot)$ and $\max(0, Z_i(\cdot))$ are continuous.
- Then $F(p)$ is continuous. Moreover $\forall_{p\in\overline{\Delta}} \forall_i F_i(p) \geq 0$ and $\sum F_i(p) = 1$ by construction.
- Then $F : \overline{\Delta} \to \overline{\Delta}$ and by Brouwer there exists $p^\star$ such that $p^\star = F(p^\star)$.
- Then:
$$p_i^\star = \frac{p_i^\star + \max(0, Z_i(p^\star))}{1 + \sum \max(0, Z_j(p^\star))} \to p_i^\star = \lambda\left(p_i^\star + \max(0, Z_i(p^\star))\right)$$
Note that $\lambda = 1$. If $\lambda < 1$ then $0 < \max(0, Z_i(p^\star))$, since this holds for all i it follows that $Z(p^\star) \geq 0$, this violates Walras' law since $p^\star \gg 0$. Then it must be $\lambda = 1$ and $0 = \max(0, Z_i(p^\star))$.
- This gives the result $Z(p^\star) \leq 0$. By Walras' law one gets that if $p^\star \in \Delta$ then $Z(p^\star) = 0$.

Theorem (-Extended- Very easy existence theorem) Let $B^l \in \mathbb{R}^l$ be compact and convex. Let $\psi : \overline{\Delta} \rightrightarrows B^l$ be an excess demand correspondence. If ψ is nonempty and convex valued, u.h.c. and satisfies Walras' law $\left(\forall_{p\in\overline{\Delta}} \forall_{Z\in\psi(p)} p \cdot Z = 0\right)$, then there exists $p^\star \in \overline{\Delta}$ and $Z^\star \in \psi(p^\star)$ such that $Z^\star \leq 0$. Moreover $Z^\star = 0$ if $p^\star \in \Delta$.

Outline:

- Note that $\bar{\Delta} \times B^l$ is nonempty, compact and convex.
- Define $\Gamma : \overline{\Delta} \times B^l \rightrightarrows \overline{\Delta} \times B^l$ such that Γ is nonempty , convex valued and u.h.c. By Kakutani Γ has a fixed point on $\overline{\Delta} \times B^l$.
- Argue that the fixed point is a competitive equilibrium.

Proof:

- Let $\Gamma(p,Z) = (F(p,Z), \psi(p))$. With $F(p,Z)$ defined as above.
- Since ψ is non-empty and convex valued, and F is a function (hence non-empty and convex valued), then we have that Γ is non-empty and convex valued.
- Since F is continuous and ψ is u.h.c. and have compact range they are closed, then Γ is closed, and since it has compact range is also u.h.c.
- By Kakutani there exists $(p^\star, Z^\star) \in \overline{\Delta} \times B^l$ such that $(p^\star, Z^\star) \in \Gamma(p^\star, Z^\star)$.
- Then $p^\star = F(p^\star, Z^\star)$ and $Z^\star \in \psi(p^\star)$. This gives $p^\star \cdot Z^\star = 0$.
- By the same argument of the VEET we get $Z^\star \leq 0$ and $Z^\star = 0$ if $p^\star \in \Delta$.

Theorem (Easy existence theorem) Let $Z : \Delta \to \mathbb{R}^l$ be an excess demand function. If Z is continuous, bounded from below, satisfies Walras' law $\left(\forall_{p\in\overline{\Delta}} p \cdot Z(p) = 0\right)$, and the boundary condition (above), then there exists $p^\star \in \Delta$ such that $Z(p^\star) = 0$.

Outline:

- Note that Z is defined for Δ and not for $\bar{\Delta}$.
- Define $\mu : \overline{\Delta} \rightrightarrows \overline{\Delta}$ such that it is nonempty , convex valued and u.h.c. By Kakutani μ has a fixed point in $\overline{\Delta}$.
- Argue that the fixed point is in Δ and that it is a competitive equilibrium.

Proof:

- Let $\mu(p) = \begin{cases} \left\{ \overline{q} \in \overline{\Delta} | \overline{q} \in \underset{q\in\overline{\Delta}}{\operatorname{argmax}}\, q \cdot Z(p) \right\} & \text{if } p \in \Delta \\ \left\{ \overline{q} \in \overline{\Delta} | \overline{q} \cdot p = 0 \right\} & \text{if } p \in \partial\Delta \end{cases}$.
- μ is nonempty and convex valued.
  - Consider $p \in \Delta$.
    - μ is given by the set of argmax of the function $q \cdot Z(p)$ for $q \in \overline{\Delta}$. The objective function is continuous and the feasible set for q is nonempty, compact and (as a correspondence) constant, hence continuous. Then by the theorem of the maximum the set of argmax (as a function of p) is nonempty, compact valued and u.h.c. (This is using the continuity assumption of $Z(p)$).
    - Consider $q, q' \in \mu(p)$ and $\lambda \in (0,1)$. Let $K = q \cdot Z(p) = q' \cdot Z(p)$. Then $\left(\lambda q + (1-\lambda) q'\right) \cdot Z(p) = K$ which implies $\lambda q + (1-\lambda) q' \in \mu(p)$. This is, $\mu(p)$ is convex valued for $p \in \Delta$.
      This also follows from ToM under convexity.

- Consider $p \in \partial\Delta$.
  * There exists $q \in \overline{\Delta}$ such that for k with $p_k > 0$ has $q_k = 0$. Thus $q \in \mu(p) \neq \emptyset$.
  * Consider $q, q' \in \mu(p)$ and $\lambda \in (0,1)$. Then $\left(\lambda q + (1-\lambda) q'\right) \cdot p = 0$ which implies $\lambda q + (1-\lambda) q' \in \mu(p)$. This is, $\mu(p)$ is convex valued for $p \in \partial\Delta$.

- In order to use Kakutani it is left to show that μ has a closed graph (or is u.h.c.). Take $p \in \overline{\Delta}$, $p_n \to p$ and $q_n \in \mu(p_n)$ such that $q_n \to q$.
  - If $p \in \Delta$.
    * Then it must be that p_n is infinitely often (i.o.) in Δ. Suppose the contrary, then there is a subsequence of p_n such that $\{p_{n_k}\} \subset \partial\Delta$, since p_n converges then $p_{n_k} \to p$, since $\partial\Delta$ is closed then $p \in \partial\Delta$ which is a contradiction.
    * Then take wlog $\{p_n\} \subset \Delta$. As noted above in this case μ is u.h.c. this implies it has a closed graph.
  - If $p \in \partial\Delta$ and $\{p_n\}$ is i.o. in $\partial\overline{\Delta}$.
    * Then consider the subsequence $\{p_{n_k}\} \subset \partial\Delta$. It holds that $q_{n_k} \cdot p_{n_k} = 0$ for all k and $q_{n_k} \to q$. Since the dot product is continuous we have $\lim q_{n_k} \cdot \lim p_{n_k} = 0$ which is $q \cdot p = 0$, then $q \in \mu(p)$. Then μ has a closed graph.
  - If $p \in \partial\Delta$ and $\{p_n\}$ is i.o. in Δ and only finitely often in $\overline{\Delta}$.
    * Then consider the subsequence $\{p_{n_k}\} \subset \Delta$. It holds that $q_{n_k} \cdot Z(p_{n_k}) = \max \overline{q} \cdot Z(p_{n_k})$ for all k and $q_{n_k} \to q$.
    * Suppose for contradiction that $q \neq \mu(p)$, then $q \cdot p > 0$. This is $\exists_j q^j, p^j > 0$.
    * Since $q_{n_k} \to q$ and $p_{n_k} \to p$ we have $q^j_{n_k} \to q^j$ and $p^j_{n_k} \to p^j$. Then there exits $\overline{k}$ such that $\forall_{k > \overline{k}} q^j_{n_k} > 0 \wedge p^j_{n_k} > 0$
    * Since $q^j_{n_k} > 0$ and $q_{n_k} \cdot Z(p_{n_k}) = \max \overline{q} \cdot Z(p_{n_k})$ it follows that $Z^j(p_{n_k}) \geq Z^i(p_{n_k})$ for all i.
    * Since $p_{n_k} \to p \in \partial\Delta$ by the boundary condition $\|Z(p_{n_k})\| \to \infty$, since $Z(\cdot)$ is bounded from below it follows that $Z^j(p_{n_k}) \to \infty$.
    * This violates Walras' law since $p^j_{n_k} > 0$ and then $p_{n_k} \cdot Z(p_{n_k}) \to \infty$.
    * Then it must be that $q \in \mu(p)$, then μ has a closed graph.

- By Kakutani μ has a fixed point in $\overline{\Delta}$. $\exists_{p^\star} p^\star \in \mu(p^\star)$.
  - $p^\star \in \Delta$. Suppose for contradiction that $p^\star \in \partial\Delta$, then $p^\star \cdot p^\star = 0$. This contradicts $p^\star \in \overline{\Delta}$.
  - Then $p^\star \cdot Z(p^\star) = \max \overline{q} \cdot Z(p^\star)$. Since $\forall_i p^{\star i} > 0$ it follows that $\forall_{i,j} Z^i(p^\star) = Z^j(p^\star) = \overline{Z}(p^\star)$.
  - By Walras' law $0 = p^\star \cdot Z(p^\star) = \sum p^{\star i} Z^i(p^\star) = \overline{Z}(p^\star) \sum p^{\star i} = \overline{Z}(p^\star)$. Which completes the proof $(Z(p^\star) = 0)$.

Theorem (-Extended- easy existence theorem) Let $\psi : \Delta \rightrightarrows B^l$ be an excess demand correspondence. If ψ is:

- Nonempty and convex valued, and u.h.c.
- Satisfies Walras' law $\left(\forall_{p\in\overline{\Delta}}\forall_{Z\in\psi(p)} p \cdot Z = 0\right)$.
- Satisfies boundedness from below condition: $\exists_{B>0}\forall_{p\in\Delta}\forall_{Z\in\psi(p)} Z \geq (-B, \ldots, -B) \in \mathbb{R}^l$.
- Satisfies boundary condition (above).

Then there exists $p^\star \in \Delta$ and $Z^\star \in \psi(p^\star)$ such that $Z^\star = 0$.

Proof:

Definition: Let $0 < \epsilon \leq 1/l$. Define a subset of the l-dimensional simplex as: $\Delta_\epsilon = \left\{p \in \Delta | \forall_{j\in\{1,\ldots,l\}} p_j \geq \epsilon\right\}$

Proposition: Let $0 < \epsilon \leq 1/l$. Δ_ϵ is nonempty, convex and compact.

Proposition: Let $\epsilon > 0$. $\psi(p)$ is bounded for any $p \in \Delta_\epsilon$.

i. $\psi(p)$ is bounded below on Δ, and $\Delta_\epsilon \subset \Delta$, then $\forall_{p\in\Delta_\epsilon}\forall_{Z\in\psi(p)}\forall_j Z_j \geq -B$.

ii. $\psi(p)$ is satisfies Walras' law on Δ, then $\forall_{p\in\Delta_\epsilon}\forall_{Z\in\psi(p)} p \cdot Z = 0 \rightarrow p_j Z_j = -\sum\limits_{k\neq j} p_k Z_k$. And
$$p_j Z_j = -\sum_{k\neq j} p_k Z_k \leq B \sum_{k\neq j} p_k \leq B$$

iii. Since $p_j \geq \epsilon$, then $Z_j \leq B/p_j \leq B/\epsilon$.

iv. If $p \in \Delta_\epsilon$ then $\psi(p) \subset [-B, B/\epsilon]^l$.

Note: $[-B, B/\epsilon]^l$ is nonempty, convex and compact for $\epsilon > 0$. Moreover $\Delta_\epsilon \times [-B, B/\epsilon]^l$ is also nonempty, convex and compact for $0 < \epsilon \leq 1/l$.

Definition: Let $0 < \epsilon \leq 1/l$. Define a correspondence $\mu_\epsilon : [-B, B/\epsilon]^l \rightrightarrows \Delta_\epsilon$ as:
$$\mu_\epsilon(Z) = \left\{p \in \Delta_\epsilon | p \in \underset{q\in\Delta_\epsilon}{\operatorname{argmax}}\, q \cdot Z\right\}$$

Proposition: μ_ϵ is u.h.c, nonempty, convex and compact valued.

i. Note that $q \cdot Z$ is continuous in q and that Δ_ϵ is a continuous and compact valued correspondence in Z, then by ToM μ_ϵ is nonempty, compact valued and u.h.c.

ii. Convexity: Let $p, p' \in \mu_\epsilon(Z)$ and $\lambda \in (0,1)$, since p and p' are both argmax of $q \cdot Z$ denote $M = p \cdot Z = p' \cdot Z$. Note that $\left(\lambda p + (1-\lambda) p'\right) \cdot Z = \lambda p \cdot Z + (1-\lambda) p' \cdot Z = \lambda M + (1-\lambda) M = M$, then $\lambda p + (1-\lambda) p' \in \mu_\epsilon$. This also follows from ToM under convexity.

Definition: Let $0 < \epsilon \leq 1/l$. Define a correspondence

$$\Gamma_\epsilon : \Delta_\epsilon \times [-B, B/\epsilon]^l \rightrightarrows \Delta_\epsilon \times [-B, B/\epsilon]^l \qquad \Gamma_\epsilon (p, Z) = (\mu_\epsilon (Z), \psi (p))$$

Γ_ϵ is u.h.c., nonempty and convex valued. (This follows from μ_ϵ and ψ being u.h.c., nonempty and convex valued).

Fixed Point: By Kakutani's fixed point theorem, Γ_ϵ has a point in $\Delta_\epsilon \times [-B, B/\epsilon]^l$.

$$\exists_{(p_\epsilon, Z_\epsilon) \in \Delta_\epsilon \times [-B, B/\epsilon]^l} (p_\epsilon, Z_\epsilon) \in \Gamma_\epsilon ((p_\epsilon, Z_\epsilon))$$

Boundedness of the fixed point: Since $(p_\epsilon, Z_\epsilon) \in \Gamma_\epsilon ((p_\epsilon, Z_\epsilon))$ we have: $p^\star_\epsilon \in \mu_\epsilon (Z^\star_\epsilon)$ and $Z^\star_\epsilon \in \psi (p^\star_\epsilon)$. The first condition implies that $\forall_{q \in \Delta_\epsilon} p_\epsilon \cdot Z_\epsilon \geq q \cdot Z_\epsilon$ and the second (by Walras' law) $p_\epsilon \cdot Z_\epsilon = 0$. Then $\forall_{q \in \Delta_\epsilon} 0 \geq q \cdot Z_\epsilon$.

Consider $q = (\epsilon, \ldots, 1 - (l-1)\epsilon)$, $q \in \Delta_\epsilon$ since $\sum q = 1$ and $\forall_j q_j \geq \epsilon$ (this follows for the last component since $1 - (l-1)\epsilon \geq \epsilon \iff \epsilon \leq 1/l$ which is always satisfied). Then:

$$0 \geq q \cdot Z_\epsilon = \sum_{j=1}^{l-1} \epsilon Z_{j\epsilon} + (1 - (l-1)\epsilon) Z_{l\epsilon} \geq \sum_{j=1}^{l-1} \epsilon (-B) + (1 - (l-1)\epsilon) Z_{l\epsilon} = -(l-1)\epsilon B + (1 - (l-1)\epsilon) Z_{l\epsilon}$$

$$Z_{l\epsilon} \leq \frac{(l-1)\epsilon}{1 - (l-1)\epsilon} B$$

This same procedure can be done for any $j \in \{1, \ldots, l\}$ then:

$$\forall_j -B \leq Z_{j\epsilon} \leq \frac{(l-1)\epsilon}{1 - (l-1)\epsilon} B \leq (l-1) B$$

Finally p_ϵ is bounded since $p_\epsilon \in \Delta_\epsilon$ which is bounded.

Equilibrium prices: There exists $p^\star \in \Delta$ and $Z^\star \in \psi (p^\star)$ such that $Z^\star = 0$.

i. Let $\{\epsilon_n\} \subset (0, 1/l]$ such that $\epsilon_n \to 0$.

ii. Define two sequences $\{p_n\} \subset \Delta$ and $\{Z_n\} \subset [-B, B/\epsilon]^l$ such that $p_n = p_{\epsilon_n}$ and $Z_n = Z_{\epsilon_n}$, a fixed point at the given ϵ_n. Then we know $Z_n \in \psi (p_n)$.

iii. Since $\{(p_n, Z_n)\} \subset \overline{\Delta} \times [-B, (l-1) B]^l$ it is a bounded sequence, hence it has a convergent subsequence $\{(p_{n_k}, Z_{n_k})\}$ on $\overline{\Delta} \times [-B, (l-1) B]^l$. $p_{n_k} \to p^\star \in \overline{\Delta}$ and $Z_{n_k} \to Z^\star$.

iv. $p^\star \in \Delta$. Suppose $p^\star \in \partial \Delta$, then by assumption, and since $Z_{n_k} \in \psi (p_{n_k})$, it holds that $\|Z_{n_k}\| \to \infty$. But for all k $Z_{n_k} \in [-B, (l-1) B]$. This is a contradiction. Then $p^\star \in \Delta$.

v. Since Z_{n_k} is a fixed point as above it holds that for all k:

$$\forall_j Z_{jn_k} \leq \frac{(l-1)\epsilon_{n_k}}{1 - (l-1)\epsilon_{n_k}} B$$

Then, since $\epsilon_{n_k} \to 0$ and $Z_{jn_k} \to Z^\star_j$ it follows that:$\forall_j Z^\star_j \leq 0$.

vi. Since ψ is u.h.c. $Z^\star \in \psi (p^\star)$.

vii. Since $Z^\star \in \psi (p^\star)$ it follows that $p^\star \cdot Z^\star = 0$. Since $p^\star \in \Delta$ and $Z^\star \leq 0$ it follows that $Z^\star = 0$.

13 Welfare

Definition (Weak Pareto dominance) An allocation $x \in \mathbb{R}^{ln}_{+}$ weakly Pareto dominates $y \in \mathbb{R}^{ln}_{+}$ if $\forall_i x_i \succeq_i y_i$ and $\exists_j x_j \succ_j y_j$.

Definition (Strong Pareto dominance) An allocation $x \in \mathbb{R}^{ln}_{+}$ strongly Pareto dominates $y \in \mathbb{R}^{ln}_{+}$ if $\forall_i x_i \succ_i y_i$.

Proposition (Equivalence of Pareto dominance) If $\succeq_i$ is continuous and monotone for all i and $\sum e_i \gg 0$ then, for $x \in \mathbb{R}^{ln}_{+}$, $\exists_{y_w \in \mathbb{R}^{ln}_{+}} y_w \text{WPD} x \iff \exists_{y_s \in \mathbb{R}^{ln}_{+}} y_s \text{SPD} x$.

Definition (Weak Pareto optimum) A *feasible* allocation $x \in \mathbb{R}^{ln}_{+}$ is weak Pareto optimal if there is no other *feasible* allocation $y \in \mathbb{R}^{ln}_{+}$ that strongly dominates x $(\forall_i y_i \succ_i x_i)$.

Definition (Strong Pareto optimum) A *feasible* allocation $x \in \mathbb{R}^{ln}_{+}$ is strong Pareto optimal if there is no other *feasible* allocation $y \in \mathbb{R}^{ln}_{+}$ that weakly dominates x $(\forall_i y_i \succeq_i x_i, \exists_j y_j \succ_j x_j)$.

Proposition (Equivalence of Pareto optimum) If $\succeq_i$ is continuous and strictly monotone for all i and $\sum e_i \gg 0$ then $x \in \mathbb{R}^{ln}_{+}$ is WPO if and only if it is SPO.

Proof: SPO to WPO is immediate. For the other direction let x be WPO, suppose it is not SPO then there exists $y \in \mathbb{R}^{ln}_{+}$ such that $(\forall_i y_i \succeq_i x_i, \exists_j y_j \succ_j x_j)$. By strict monotonicity $y_j \geq 0$, let $b \in \mathbb{R}^l$ such that $b^k = 1$ if $y_j^k > 0$ and 0 otherwise. By continuity of preferences there exists $\epsilon > 0$ such that $z_j = y_j^k - \epsilon b$ satisfies $z_j \succ_i x_j$. Define $z_i = y_i + \frac{1}{n-1}\epsilon b$, by strict monotonicity $z_i \succ_i y_i$ and then $z_i \succ_i x_i$. Finally note that z is feasible since

$$\sum_{i=1}^{n} z_i = \sum_{i=1}^{n} y_i + \sum_{i \neq j} \frac{1}{n-1}\epsilon b - \epsilon b = \sum_{i=1}^{n} y_i = \sum_{i=1}^{n} e_i$$

Then z strongly Pareto dominates x contradicting that x is a WPO. Then it must be that x is a SPO.

Note: If $x \in \mathbb{R}^{ln}_{++}$ it suffices to assume that $\succeq_i$ is monotone for all i. If $x \in \mathbb{R}^{ln}_{+}$ with some element in the boundary the proof for the proposition of equivalence between Pareto dominance needs strict monotonicity to ensure feasibility.

Note: If preferences are only monotone a counterexample can be constructed with a WPO allocation in the boundary. ($x \in \mathbb{R}^{ln}_{+}$ with $x^k = 0$ for some k). If one agent only cares about the consumption of one good, and the other is indifferent between consuming more or less of the good.

(Keler) Consider $u_1(x^1, x^2) = x^1$, $u_2(x^1, x^2) = \min(x^1, x^2)$ with aggregate endowments $E = (6, 3)$ and the allocation $x = (x_1, x_2) = ((0, 2), (3, 4))$. x is a WPO since there is no *feasible* allocation that makes trader 2 strictly better off, but it is not a SPO since the allocation $x' = ((0, 3), (3, 3))$ leaves trader 2 indifferent and makes trader 1 strictly better off.

Theorem (First welfare theorem) Let $\mathcal{E} = \{(\succeq_i, e_i, X_i)_{i=1}^n\}$ be an exchange economy with $\succeq_i$ a continuous complete preorder and $e_i \in \mathbb{R}_+^l$ for all i, moreover $\sum e_i \gg 0$. Then, if $\succeq_i$ are locally non-satiated on X_i and $(p^\star, x^\star) \in \overline{\Delta} \times \mathbb{R}_+^{ln}$ is a competitive equilibrium, $x^\star$ is a (strong) Pareto optimum.

Proof:

- Let $(p^\star, x^\star)$ be a competitive equilibrium for $\mathcal{E}$.
- Suppose for contradiction that $x^\star$ is not Pareto optimum. Then there exists $y \in \mathbb{R}^{ln}$ such that is feasible ($\sum y_i = \sum e_i$) and $\forall_i y_i \succeq_i x_i^\star, \exists_j y_j \succ_j x_j^\star$.
- Note that $\forall_i p^\star \cdot x_i^\star = p^\star \cdot e_i$. Suppose not, then $\exists_i p^\star \cdot x_i^\star < p^\star \cdot e_i$ (since $x_i^\star$ has to be affordable), then, by l.n.s. there exists $x_i^{'}$ such that $x_i^{'} \succ_i x_i^\star$ and $p^\star \cdot x_i^{'} < p^\star \cdot e_i$. This contradicts $x_i^\star$ being optimal for trader i.
- Note then that $\forall_i p^\star \cdot y_i \geq p^\star \cdot e_i$. Suppose not, then $\exists_i p^\star \cdot y_i < p^\star \cdot e_i$, by l.n.s. there exits $y_i^{'}$ such that $y_i^{'} \succ_i y_i$ and $p^\star \cdot y_i^{'} < p^\star \cdot e_i$. That is, $y^{'}$ is affordable at prices $p^\star$ for trader i. By transitivity $y_i^{'} \succ y_i \succeq x_i^\star$ which contradicts $x_i^\star$ being optimal for trader i.
- Note now that $p^\star \cdot y_j > p^\star \cdot e_j$. Suppose not, then y_j would be affordable for trader j, since $y_j \succ_j x_j^\star$ this contradicts $x_j^\star$ being optimal for trader j.
- Then $\sum p^\star \cdot y_i > \sum p^\star \cdot e_i$ which implies: $p^\star \cdot [\sum (y_i - e_i)] > 0$. Since $p^\star \in \overline{\Delta}$ this implies $\exists_k \sum \left(y_i^k - e_i^k\right) > 0$ which contradicts feasibility of y.

Theorem (Second welfare theorem) Let $\overline{x} \in \mathbb{R}_{++}^{ln}$ be a (strong) Pareto optimum for an exchange economy $\mathcal{E} = \{(\succeq_i, e_i)_{i=1}^n\}$ with $\succeq_i$ continuous, strictly monotone and strictly convex and $\sum e_i \gg 0$. Then $\overline{x}$ is a competitive equilibrium allocation for $\mathcal{E} = \{(\succeq_i, \overline{e}_i)_{i=1}^n\}$ with $\overline{e}_i = \overline{x}_i$. Moreover, $\overline{x}$ is the only competitive equilibrium allocation and the price vector $(p^\star)$ satisfies $p^\star \geq 0$ and $p^\star \neq 0$.

Note: Strict monotonicity can be weakened to local non-satiation. Strict convexity is only needed for $\overline{x}$ to be the only equilibrium allocation, it can be weakened to convexity.

Note: If $\mathcal{E}$ is defined with utility functions (u_i) instead of preferences $(\succeq_i)$ the strict convexity assumption translates to strict quasi-concaveness of u_i. That is, that the upper contours are strictly convex.

Proof:

- Let $U_i(\overline{x}_i) = \left\{x \in \mathbb{R}_+^l | x \succ_i \overline{x}_i\right\}$ be the (relatively) open contour of trader i at $\overline{x}_i$. Since preferences are convex U_i is convex for all i. Define $U(\overline{x}) = \sum U_i(\overline{x}_i)$. This set is open and convex.
- Define $E = \sum \overline{x}_i$. Since $\overline{x}$ is P.O. E equals the aggregate endowments of economy $\mathcal{E}$.
  - Note that $E \notin U(\overline{x})$. Suppose it does, then $\forall_i \exists_{x_i} x_i \in U_i(\overline{x}_i) \wedge \sum x_i = E$. Then $\forall_i x_i \succ_i \overline{x}_i$, since E is feasible, this contradicts $\overline{x}$ being P.O.

- Then, by the separating hyperplane theorem, there exists $p \in \mathbb{R}^l \backslash \{0\}$ such that $\forall_{x \in U(\overline{x})} p \cdot (x - E) \geq 0$.

  - Note that $p \geq 0$. Denote b_j the j^{th} element of the unit basis of $\mathbb{R}^l$. Take $x_i = \overline{x}_i + \frac{1}{n} b_j$, since $x_i \geq \overline{x}_i$ and $x_i \neq \overline{x}_i$ we have, by strict monotonicity $x_i \in U_i(\overline{x}_i)$. Then $x = \sum x_i = E + b_j \in U(\overline{x})$. Then $p \cdot (x - E) = p \cdot b_j = p_j \geq 0$. Since this is true for all j we have $p \geq 0$.

- If $x_i \succ_i \overline{x}_i$ then $p \cdot x_i \geq p \cdot \overline{x}_i$:

  - Let $x_i \succ \overline{x}_i$. By continuity of preferences: $\exists_{\epsilon > 0} \tilde{x}_i = x_i - \epsilon b \succ_i \overline{x}_i$. $\tilde{x}_i \in U_i(\overline{x}_i)$.
  - By monotonicity for $j \neq i$ we have: $\tilde{x}_j = \overline{x}_j + \frac{\epsilon}{n-1} b \succ_j \overline{x}_j$. $\tilde{x}_j \in U_j(\overline{x}_j)$.
  - Then, $\sum \tilde{x}_i \in U(\overline{x})$ and by separating hyperplane: $p \cdot [\sum (\tilde{x}_i - \overline{x}_i)] \geq 0$ this is: $p \cdot [\tilde{x}_j - \overline{x}_j] \geq 0$ the desired result.

- Moreover, if $x_i \succ_i \overline{x}_i$ then $p \cdot x_i > p \cdot \overline{x}_i$:

  - Let $x_i \succ \overline{x}_i$ and suppose for contradiction that $p \cdot x_i = p \cdot \overline{x}_i$.
  - By continuity $\exists_{\lambda \in (0,1)} \lambda x_i \succ_i \overline{x}_i$. By previous result: $p \cdot (\lambda x_i) \geq p \cdot \overline{x}_i$.
  - Since $\lambda < 1$ we have $p \cdot (\lambda x_i) < p \cdot x_i = p \cdot \overline{x}_i$. This is a contradiction.

- Then, at prices p and endowment $\overline{x}_i$, $\overline{x}_i$ is in trader's i demand $(\forall_x p \cdot x \leq p \cdot \overline{x}_i \rightarrow \overline{x}_i \succeq_i x)$. Then $\overline{x}$ is a competitive equilibrium allocation at price p.

  - Since preferences are strictly convex demand is single valued. Then $\overline{x}$ is the only equilibrium allocation at price p.

Definition (Economy with transfers) Let $\mathcal{E} = \{(\succeq_i, e_i, X_i,)_{i=1}^n\}$ be an economy. An economy with transfers $\mathcal{E}_T = \{(\succeq_i, e_i, X_i, T_i)_{i=1}^n\}$ with $\sum_{i=1}^n T_i = 0$, differs from $\mathcal{E}$ only in the definition of the budget set:

$$B_i(p, e_i, T_i) = \{x \in X_i | p \cdot x \leq p \cdot e_i + T_i\}$$

Proposition (Implementation of PO with transfers) Let $\hat{x} \in PO$ of economy $\mathcal{E} = \{(\succeq_i, e_i, X_i,)_{i=1}^n\}$, then if $\hat{x}$ is implemented as a CE allocation of the economy $\hat{\mathcal{E}} = \{(\succeq_i, \hat{x}_i, X_i,)_{i=1}^n\}$ under price $\hat{p}$, it is implemented as a CE allocation of the economy $\mathcal{E}_T = \{(\succeq_i, e_i, X_i, T_i)_{i=1}^n\}$ under price $\hat{p}$, where $T_i = \hat{p} \cdot \hat{x}_i - \hat{p} \cdot e_i$.

Proof: First note that $\sum_i T_i = p \cdot (\sum_i x_i - \sum_i e_i) = 0$, then transfers are balanced. At price $\hat{p}$ it follows, for all agent i and all $y \in X_i$ such that $\hat{p} \cdot y \leq \hat{p} \cdot e_i + T_i$, that $\hat{p} \cdot y \leq \hat{p} \cdot \hat{x}_i$. Since $(\hat{x}, \hat{p})$ is a CE of $\hat{\mathcal{E}}$ it follows that $y \preceq_i \hat{x}_i$, then $(\hat{x}, \hat{p})$ is a CE of $\mathcal{E}_T$.

Proposition (SWT with non-convex consumption sets) Let $\mathcal{E} = \{(\succeq_i, e_i, X_i)_{i=1}^n\}$ be an economy with $\succeq_i$ strictly monotone, strictly convex and continuous preferences defined on $\mathrm{co}\,(X_i)$. Let $Q = \{x \in X | x \text{ is Pareto Optimal}\}$ then if $\mathcal{E}' = \{(\succeq_i, e_i, \mathrm{co}\,(X_i))_{i=1}^n\}$ with set of Pareto Optimal Allocations Q' is such that $Q \subseteq Q'$ then the Second Welfare Theorem holds in $\mathcal{E}$.

Proof: By construction the Second Welfare Theorem applies to $\mathcal{E}'$. Suppose it does not apply to $\mathcal{E}$, this implies $\exists x \in Q$ such that it is not a CE allocation when $\hat{e} = x$. Then for any p, there exists an agent j and $y_j \in X_j$ such that $y_j \succ x_j$ and $p \cdot y_j \leq p \cdot \hat{e}_j$. Since $y_j \in X_j$ it must be that $y_j \in \mathrm{co}\,(X_j)$ meaning it was available in $\mathcal{E}'$. But $x \in Q$ implies $x \in Q^{'}$then there exists a price $\hat{p}$ such that x is a CE in $\mathcal{E}'$, by the argument above for that price there exists an agent j and a bundle $y_j \in X_j \subseteq \mathrm{co}\,(X_j)$ such that $y_j \succ x_j$ and $\hat{p} \cdot y_j \leq \hat{p} \cdot \hat{e}_j$ which contradicts that x being a CE when $\hat{e} = x$ in $\mathcal{E}^{'}$, hence contradicting the second welfare theorem in $\mathcal{E}^{'}$.
This contradiction implies that the second welfare theorem applies to $\mathcal{E}$.

Corollary:

i. If $x \in Q \cap Q^{'}$ then the same argument above applies, then it follows that any Pareto Optimum allocation that is a PO in both economies can be implemented in the non-convex economy, even if the inclusion condition ($Q \subseteq Q'$) does not hold.

ii. If $x \in \mathrm{int}\,(X) \cap Q$ then it follows that $x \in Q \cap Q^{'}$ which implies that all PO allocation in the interior of the non-convex economy are implementable as CE allocations in the sense of the Second Welfare Theorem.

Corollary (Implementation of PO in non-convex economy) Let $\hat{x} \in Q \cap Q^{'}$, then $\hat{x}$ is implemented as a CE allocation of the economy $\hat{\mathcal{E}} = \{(\succeq_i, \hat{x}_i, X_i)_{i=1}^n\}$ with the same price that implements it under $\hat{\mathcal{E}}^{'} = \{(\succeq_i, \hat{x}_i, \mathrm{co}\,(X_i))_{i=1}^n\}$.

Proof: Let $\hat{x} \in Q \cap Q^{'}$ then, by the SWT applied to economy $\mathcal{E}^{'}$ there exists $\hat{p}$ such that $(\hat{x}, \hat{p})$ is a CE of $\hat{\mathcal{E}}^{'}$, in particular for all agent i it holds that for all $y^{'} \in \mathrm{co}\,(X_i)$ if $\hat{p} \cdot y^{'} \leq \hat{p} \cdot \hat{x}_i$ then $y^{'} \preceq_i \hat{x}_i$. Note that for if $y \in X_i$ then $y \in \mathrm{co}\,(X_i)$, then for all $y \in X_i$ such that $\hat{p} \cdot y \leq \hat{p} \cdot \hat{x}_i$ it follows that $y \preceq_i \hat{x}_i$. Then $(\hat{x}, \hat{p})$ is a CE of $\hat{\mathcal{E}} = \{(\succeq_i, \hat{x}_i, X_i)_{i=1}^n\}$.

14 Core and Competitive Equilibrium

Definition (Individually rational allocation) An allocation $x \in \mathbb{R}_+^{ln}$ is individually rational in exchange economy $\mathcal{E}$ if $\forall_i x_i \succeq_i e_i$.

Definition (Coalition) A coalition S is a subset of the traders in exchange economy $\mathcal{E}$. Number of coalitions $2^n - 1$. Includes coalition of all traders and coalition of a single trader.

Definition (Blocked allocation - Debreu and Scarf (1963)) An allocation is blocked by a coalition S with s traders if there exits $y \in \mathbb{R}_+^{ls}$ such that is feasible for the coalition $\left(\sum\limits_{i \in S} y_i = \sum\limits_{i \in S} e_i\right)$ and $\forall_{i \in S} y_i \succeq_i x_i$ and $\exists_{j \in S} y_j \succ_j x_j$.

Note: Under strict monotonicity and continuous preferences an allocation is blocked by a coalition S if $\forall_{i \in S} y_i \succ_i x_i$ and y is feasible for S.

Definition (Core) The core of exchange economy $\mathcal{E}$ is the set of allocations that cannot be blocked by any coalition.

Proposition (Core and Pareto optimum) The core of economy $\mathcal{E}$ is contained in the set of SPO allocations. Consider allocation of all traders.

Proposition (Core and Competitive Equilibrium) The set of CE allocations is contained in the core. Proof is identical to FWT applied to the blocking coalition.

Note: Sufficient conditions for non-emptiness of the core are those that guarantee existence of equilibrium.

Definition (R-Replica economy) The R replica of economy $\mathcal{E}$, noted $\mathcal{E}^R$, is an economy with Rn traders, with n types and R identical traders of each type. $\mathcal{E}^R = \left\{((\succeq_{ir}, e_{ir})_{i=1}^n)_{r=1}^R\right\}$ with $\succeq_{ir} = \succeq_{ir'}$ and $e_{ir} = e_{ir'}$ for all $r, r' \in \{1, \dots R\}$.

Proposition (Equal treatment property) Let $\mathcal{E}$ be an exchange economy and $\mathcal{E}^R$ its R replica. If $x \in \text{Core}\left(\mathcal{E}^R\right)$ then $x_{ir} = x_{ir'}$ for all $r, r' \in \{1, \dots R\}$.

Proof:

- Let $x \in \text{Core}\left(\mathcal{E}^R\right)$ such that there exists $j \in \{1, \dots, n\}$ and $r, r' \in \{1, \dots R\}$ for which $x_{jr} \neq x_{jr'}$.
- Consider a coalition S formed by one agent of each type.
- Let $\underline{x}_i \in \{x_{i1}, \dots, x_{ik}\}$ such that $\forall_k x_{ik} \succeq_i \underline{x}_i$. Coalition S is formed by the agents with allocations $\underline{x}_i$.
- Define an allocation $x_i^S = \frac{1}{R} \sum\limits_{k=1}^{R} x_{ik}$, by strict convexity $x_i^S \succeq_i \underline{x}_i$ for all i and $x_j^S \succ_j \underline{x}_j$.

- If allocation x^S is feasible S would then be a blocking coalition, recalling that x is feasible:
$$\sum_{i=1}^{n} x_i^S = \frac{1}{R}\sum_{i=1}^{n}\sum_{k=1}^{R} x_{ik} = \frac{1}{R}\left(R\sum_{i=1}^{n} e_i\right) = \sum_{i=1}^{n} e_i$$
- Since there is always a blocking coalition for x it must be that the equal treatment property holds.

Theorem (Limit of core and CE Debreu and Scarf (1963)) Let $\mathcal{E}$ be an exchange economy with $\succeq_i$ continuous, l.n.s and strictly convex, and $e_i \gg 0$. and $\mathcal{E}^R$ its R replica. Then $\mathrm{CE}(\mathcal{E}) = \bigcap_{R=1}^{\infty} \mathrm{Core}\left(\mathcal{E}^R\right)$. In other words, if an allocation is in the core of all replica economies then it is a CE allocation.

Note: It is required in the Debreu-Scarf article that $e_i \gg 0$, this is not the usual condition $\sum e_i \gg 0$.

15 Non-Convexities

Existence results, the second welfare theorem and the Debreu-Scarf theorem (core) assume convexity of preferences and consumption sets of all traders. These assumptions imply some "consistency" on preferences and choice sets, in the sense that they impose behavioral restrictions on the agents and certain "completeness"-like property in the choice sets. Intuitively convexity allows to compare and move along the commodity space in a systematic way, maintaining certain order in preferences. In this way convex preferences can be seen as an strengthening of the rationality assumption.
The Theorem of Maximum is useful to understand the effect of non-convexities in preferences and choice sets on individual excess demand, these effects translate to aggregate excess demand under a finite number of traders.

i. Non-convex choice sets:

- The budget correspondence may fail to be compact valued. Then Weierstrass theorem does not apply and (individual) demand might fail to be non-empty.
- The budget correspondence may fail to be continuous. The ToM would not apply to the UMP and the (individual) demand might fail to be u.h.c.
- The budget correspondence may fail to be convex valued. The ToM under convexity would not apply to the UMP and the (individual) demand might fail to be convex valued.
- The preferences may fail to be locally non-satiated in the consumption set (for example with indivisible goods). This prevents the FWT to apply.

ii. Non-convex preferences:

- Under preferences that are not strictly convex demand might fail to be single valued for all prices.
- Under non-convex preferences the utility function ceases to be quasi-concave. The ToM under convexity would not apply to the UMP and the (individual) demand might fail to be convex valued.

Upper hemi-continuity and convex valuedness of the excess demand correspondence are used actively in the proofs of the existence theorems, second welfare theorem and Debreu-Scarf theorem. They are needed to establish the existence trough Kakutani's (or Brouwer's) fixed point theorem, that requires both properties, and trough the separating hyperplane theorem, that requires convexity of upper contour sets. Non-emptiness of the demand correspondence invalidates all of the results.

Under non-convex preferences it is possible to define a quasi equilibrium, provided that the number of traders in the economy is large enough. This equilibrium is called ϵ-equilibrium in Starr (1969). The idea is that individual non-convexities in demand can be dealt with by introducing more agents to the economy, the result (for finitely many traders) is not necessarily a CE but can be shown to be arbitrarily close to one.

When the economy is composed by a continuum of agents another answer is provided to non-convexity issues.

16 Continuum of Agents

An alternative framework for modeling perfectly competitive economies is to assume that agents are atomless elements in a continuum. This approach treats directly the idea of an individual agent having a negligible effect on the market. Results under these conditions are less dependent on convexity assumptions.

Definition (Exchange economy with a continuum of agents) An exchange economy (with l goods and traders in $T = [0,1]$) is formed by pairs of preferences and endowments that characterize traders (indexed by i). Then an economy is $\mathcal{E} = \left\{ (\succ_i, e(i))_{i \in T} \right\}$, where $\succ_i$ is a relation on $\mathbb{R}^l_+$ that represents trader i's preferences, it is assumed that $\int_T e(i)\, di > 0$.

Definition (Allocation and feasibility) An allocation is a function $x : T \to \mathbb{R}^l_+$ such that each coordinate is Lebesgue integrable over T. $e(i)$ is the function of initial endowments. An allocation is feasible if $\int_T x = \int_T e$.

Definition (Strong Pareto Optima) A feasible allocation x is (Strong) Pareto optimum if there does not exists a set $S \subseteq T$ and a feasible allocation y such that $\mu(S) > 0$, $y(i) \succeq_i x(i)$ for almost every i and $y(j) \succ_j x(j)$ for all $j \in S$.

Definition (Weak Pareto Optima) A feasible allocation is weak Pareto optimum if there does not exists a feasible allocation y such that $y(i) \succ_i x(i)$ for almost all $i \in T$.

Definition (Coalition) A coalition of traders is a Lebesgue measurable subset of T. If it is of measure 0, it is called null.

Definition (Core) The core is the set of all allocations that are not dominated via any non-null coalition. An allocation y is dominated by x via coalition S if $\forall_{i \in S} x(i) \succ_i y(i)$ and $\int_S x(i) = \int_S e(i)$.

Note: Under strictly monotone and continuous preferences this definition is equivalent to one in which a dominated coalition is such that $\forall_{i \in T} x(i) \succeq_i y(i)$, $\int_S x(i) = \int_S e(i)$ and there exits $S' \subseteq S$ such that $\mu(S) > 0$ and $\forall_{j \in S'} x(j) \succ_j y(j)$.

Definition (Competitive Equilibrium) A competitive equilibrium is $\{p, x\}$, a pair of a price vector p and an allocation x, such that for almost every trader i, $x(i) = x_i(p, \succ_i, e(i))$. An allocation is feasible by definition in this environment.

Proposition (Equivalence of Pareto optima) If preferences are strictly monotone and continuous then an allocation x is a strong Pareto optimum if and only if it is a weak Pareto optimum.

Theorem (First welfare theorem with a continuum of agents) If preferences $\succeq_i$ are locally non-satiated then all competitive equilibrium allocation x is Pareto optimum.

Proof: Suppose not, and let p be a competitive equilibrium price associated with x. Then for almost all $j \in S$ it must be that $p \cdot y(j) > p \cdot e(j)$ (or else $x(j)$ wouldn't be maximal with respect to $\succeq_j$) and for almost all $i \in T$: $p \cdot y(i) \geq p \cdot e(i)$ (using local non-satiation for obtaining z such that $p \cdot z < p \cdot e(i)$ and $z \succ_i x(i)$). Then integrating gives:

$$\int_S p \cdot y(j)\, dj > \int_S p \cdot e(j)\, dj \qquad \int_{T \setminus S} p \cdot y(i)\, di \geq \int_{T \setminus S} p \cdot e(i)\, di$$

And adding:

$$\int_T p \cdot y(i)\, di > \int_T p \cdot e(i)\, di \longrightarrow p \cdot \int_T (y(i) - e(i))\, di > 0$$

which contradicts feasibility of allocation y.

Theorem (Core and competitive equilibrium - Aumann (1964)) If preferences are strictly monotone, continuous and measurable (open contour sets are Lebesgue measurable in T), then the core coincides with the set of competitive equilibrium.

Note: Aumann's result does not require preferences to be complete transitive or, in particular, convex.

Note: The Debreu-Scarf limiting result and Starr's result on non-convexities are obtained in this "large" economy. Note that preferences are not asked to be convex.

Theorem (Existence of competitive equilibrium - Aumann (1966)) If preferences $\succeq_i$ are complete, transitive and reflexive, and (as before) strictly monotone, continuous and measurable (open contour sets are Lebesgue measurable in T), then there exists a competitive equilibrium.

Theorem (Existence of CE with non-convex consumption sets - Yamazaki (1978)) If, moreover, endowment distribution is disperse and choice sets are non-convex, then there exists a competitive equilibrium for economy $\mathcal{E}$.

Note: The competitive equilibrium considered by Yamazaki allows for free disposal, in the sense that the equilibrium allocation is such that $\int_T x(i) \leq \int_T e(i)$.

Definition (Dispersed Endowments) The endowment distribution is said to be dispersed when the distribution of wealth ($\mu_{p,e}(B) = \mu\{i \in T | p \cdot e(i) \in B\}$ where μ is the Lebesgue measure and B is a Borel set of $\mathbb{R}$) is absolutely continuous for every price (interpret this as the CDF of wealth being continuous which implies that wealth has a PDF with no mass points). That implies that the distribution of individual agents according to the wealth they own does not give positive weight to any particular amount ($w = p \cdot e(i)$).

17 Infinitely Many Commodities

All the economies that have been considered so far have finitely many commodities, yet there are (interesting situations) for which the number of commodities is infinite (but countably so). There are three main cases that give rise to infinitely many commodities:

i. Time. When trade takes place in subsequent periods infinitely far into the future. Thus the economy includes the goods traded at each period, indexed by time.

ii. Uncertainty. When trade involves products that depend on realization of states of nature, and there are infinitely many possible states. A single type of good, like a "state contingent" asset, includes all the infinitely many goods that arise from state differentiation.

iii. Product differentiation. When trade involves products that are closely related, a same type of commodity has infinitely many products related to it.

Considering infinitely many commodities has profound effects over the mathematical results used so far. In the space $\mathbb{R}^{\infty}$ the equivalence between compact and closed and bounded sets does not hold. This invalidates the proofs that rely on compactness and fixed points theorems when establishing the existence of the equilibrium. It also posses questions over the definition of the objects that define an economy.

i. What is the set of commodities?

ii. What is the consumption set?

iii. What is the commodity space?

iv. What does it mean for preferences (or utilities) to be continuous?

v. How to define monotonicity properties when there are infinitely many goods?

vi. What type of preferences can be represented by a utility function?

vii. What is the price space?

These questions had natural solutions when the number of commodities was finite, but are harder to solve in the current setting, they also involve the selection of a proper topology and metric for the spaces under consideration. Depending on the topology used the definition of compact set might be easy, but the set of continuous functions very restricted. Note that open sets are used to defined continuity while closed and compact sets for maximization. This induces a tradeoff in the selection of the topology.

Some examples of these type of economies are:

i. OLG models. (Depending on the definition of the commodity space the prices are different).

ii. Infinitely lived consumers.

iii. Choice under uncertainty among lotteries with infinitely many outcomes.

iv. Product Differentiation. Crowding of goods, and new goods that are closed substitutes.

18 Properties of Excess Demand Functions

[This discussion follows closely the one presented in Kirman (1992) and Sonnenschein (1973)]

The question over the properties of excess demand is first a question over which properties (if any) does the general equilibrium framework imply over the aggregate demand, what does it imply for the aggregate the hypothesis of utility maximizing individuals. Second is a question over which conditions on these individuals must be imposed to get some "desired" properties on the aggregate, as a unique equilibrium, or a continuous (differentiable) relation of equilibrium outcome to "parameters" in the economy.

The relevance of the question is twofold, it informs about the implications of the framework at hand, making it testable (as with the characteristics of the Slutsky matrix in individual choice theory, Mas-Colell et al. (1995, Ch. 2.3)), and determines conditions for the use of the framework, for example, in order to answer some questions (comparative statics) it is needed to have a unique (or at leas finite set) equilibrium.

The answer to the question on the properties of aggregate excess demand is strikingly short, and is due (among others) to Sonnenschein, Mantel and Debreu. The latter showed that any continuous function satisfying Walras' law and homogeneity of degree zero can be represented (up to a compact subset of the simplex) by the aggregate excess demand function of an economy where agents have preferences represented by continuous complete pre-orders, strictly convex and monotone.

Even with all these conditions over consumers only three (basic) properties translate from individual to aggregate excess demand. Any other property cannot be expected of any given economy, for example, to have a unique equilibrium or to satisfy the weak axiom of revealed preferences (that is, that the aggregate behaves like a maximizer individual). Sonnenschein (1973) points out:

> "The present results point to the conclusion that Walras' Identity and Continuity summarize all of the restrictions on the community excess demand function which follow from the hypothesis that consumers maximize utility and producers maximize profit."

Kirman (1992), talking about the representative agent, stresses that there is no microfundation (a general condition on maximizing individuals) that would guarantee an aggregate excess demand that induce a unique or stable equilibrium:

> "The simple answer would be to find conditions implied by assumptions on the individuals in an economy which guarantee uniqueness and stability. However, a series of results starting with those of Sonnenschein (1972) and Debreu (1974) show unequivocally that no such conditions exist."

Imposing a representative agent is too restrictive on the economy being modeled.

This fact also explains why those are the only properties over which existence and other results are built, again Sonnenschein (1973):

> "Beyond Walras' Identity and Continuity, that literature [on the existence and stability of competitive equilibrium] makes no use of the fact that community demand is derived by summing the maximizing actions of agents."

A partial response to this problem (lack of further properties) is given by Debreu (1970, 1972, 1976) using methods of differential topology. Imposing further conditions on individual agents, aggregate excess demand can be guaranteed to be smooth (continuously differentiable). The conditions needed are those of a smooth economy. Debreu shows that on a family of smooth economies (for example indexed by endowments) the set of economies with finitely many equilibria is of full measure (they are generic), and that the set of equilibria depends continuously on the characteristics of the economy for these economies.

The results obtained with smooth economies make equilibrium theory useful in the sense that its implications are testable and provide insight into the effect of changes in the environment over the outcomes of the economy. Debreu shows that for smooth economies the set of economies with a finite number of equilibria is of full measure, and that the equilibrium outcome (prices and quantities) varies in a continuous (even differentiable) manner with respect to changes in the economy. This property of having locally unique and stable equilibria makes the theory useful.

Theorem (Sonnenschein, Mantel, Debreu) Let $f : \mathbb{R}^l_+ \to \mathbb{R}^l$ be a continuous function satisfying Walras' law and homogeneity of degree zero. Then for all $\epsilon > 0$ there exists an economy $\mathcal{E} = \{(\succeq_i, e_i)_{i=1}^n\}$ with $\succeq_i$ a continuous, monotone and strictly convex complete preorder and $e_i \in \mathbb{R}^l_+$, such that its aggregate excess demand $Z(p) = f(p)$ for $p \in \Delta_\epsilon$.

Definition (Smooth economy) Let $\mathcal{E} = \{(f_i, e_i)_{i=1}^n\}$ with $f_i : \overline{\Delta} \times \mathbb{R}^l_+ \to \mathbb{R}^l_+$ a demand function for each trader be an economy (that is $\forall_i \forall_{p,e_i} p \cdot f_i(p, e_i) = p \cdot e_i$ and f_i is homogenous of degree zero in prices). $\mathcal{E}$ is smooth if, for all traders i, $f_i \in C^1$. The notion of smoothness can be strengthened with the boundary condition below:

$$\forall_i \forall_{e_i} \forall_{\{p_n\} \subset \Delta} p_n \to p \in \partial\Delta \implies \|f_i(p_n, e_i)\| \to \infty$$

The above conditions can be obtained if agents have preferences that can be represented by C^2 utility functions (u) that are strictly differentiable monotone $(\forall_x D(x) \gg 0)$ and strictly differentiable concave ($\forall_x D^2(x)$ is negative definite), and satisfy the following boundary condition:

$$\forall_{x \in X} \operatorname{cl}\left\{y \in \mathbb{R}^l_{++} | u(y) \geq u(x)\right\} \bigcap \partial\mathbb{R}^l_+ = \emptyset$$

Definition (Regular economy - Debreu (1976)) Consider a family of smooth economies $\mathbb{E} = \{\mathcal{E}_i\}$ where economies are indexed by some parameter (as endowments). A regular economy is such that:

i. It is generic. Its complement (the set of critical economies) is of zero (Lebesgue) measure.

ii. Every regular economy has a discrete set of equilibria.

iii. In a neighborhood of a regular economy, the set of equilibria depends continuously on the economy.

If the excess demand function of an economy $\mathcal{E}_i$ satisfy the boundary condition (unboundedness when price approaches the boundary) these conditions are strengthened to:

i. It is generic. Its complement (the set of critical economies) is of zero (Lebesgue) measure and is closed.

ii. Every regular economy has a finite set of equilibria.

iii. In a neighborhood of a regular economy, the set of equilibria depends in a continuously differentiable manner on the economy.

Theorem (Finite set of Equilibrium Prices - Debreu (1970)) Consider an economy $\mathcal{E}(e)$ with n agents indexed by i and strictly positive endowments $e_i \gg 0$.

If for al i the excess demand function $Z_i : \Delta \to \mathbb{R}^l_+$ is continuously differentiable, and there exists j such that Z_j satisfies the boundary condition, then the set of endowments for which the economy has infinitely many equilibrium prices is a closed set with zero measure.

Then the set of endowments for which the economy has finitely many equilibrium prices is generic (has full measure).

Moreover, if for all i the excess demand function Z_i satisfies the boundary condition the set of equilibrium prices is non-empty.

Note: Sufficient conditions on utilities for these results are given by Katzner (1968): u_i is continuous in $\mathbb{R}^l_+$ and twice continuously differentiable in $\mathbb{R}^l_{++}$, is strictly monotone and strictly concave. Debreu (1972) establishes sufficient conditions over preferences.

Theorem (Sard) Let $U \subset \mathbb{R}^n$ be open and $F : U \to \mathbb{R}^m$ be C^1. Let Cr be the set of critical points of F ($\mathrm{Cr} = \{x \in U | \mathrm{rank} DF(x) < m\}$). Then the set of critical values $F(\mathrm{Cr})$ has (Lebesgue) measure zero in $\mathbb{R}^m$.

Note: When $n = m$ the condition for a critical value reduces to $\det DF(x) = 0$.

Theorem (Regular value) Let A be an n-dimensional smooth manifold, B an m- dimensional smooth manifold and $F : A \to B$ a mapping. If $y \in B$ is a regular value of F then the pre-image $F^{-1}(y)$ is a smooth manifold of dimension $n - m$ or is empty.

19 Production Economies

Definition (Production economy) A production economy (with n traders, J firms and l goods) is formed by n three tuples of preferences, endowments and shares that characterize traders (indexed by i), and J production sets that characterize the firms. Then an economy is $\mathcal{E} = \left\{ \left(\succeq_i, e_i, \{\theta_{ij}\}_{j=1}^J \right)_{i=1}^n, \{Y_j\}_{j=1}^J \right\}$, where $\succeq_i$ is a complete preorder that represents trader i's preferences, $e_i \in \mathbb{R}_+^l$ is the vector of initial endowments of that trader, $\theta_{ij} \in [0,1]$ is the share of that trader in the j^{th} firm $\left(\forall_j \sum_i \theta_{ij} = 1 \right)$ and Y_j is a non-empty and closed production set.

Note: More generally the choice set X_i of each trader should be included in the definition of the economy. Since this is usually $\mathbb{R}_+^l$ it is omitted. Alternatively the economy can be defined with u_i, a utility function that represents trader i's preferences. It is also assumed that $\sum_{i=1}^n e_i \gg 0$, this is, that there are not irrelevant goods.

Definition (Budget correspondence) The budget correspondence $B_i : \Delta \rightrightarrows \mathbb{R}_+^l$ is:

$$B_i(p) = \left\{ x \in X_i | p \cdot x \leq p \cdot e_i + \sum_{j=1}^J \theta_{ij} \pi_j \right\}$$

where π_j represents the profits generated by firm j.

Definition (Demand correspondence) The demand correspondence of trader i is:

$$x_i(p) = \left\{ x_i \in B_i(p) | \forall_{x' \in B_i(p)} x_i \succeq_i x' \right\}$$

Definition (Feasible allocation) An allocation $\left\{ (x_i)_{i=1}^n, (y_i)_{j=1}^J \right\} \in \mathbb{R}_+^{ln} \times \mathbb{R}^{lJ}$ is feasible if $\forall_i x_i \in X_i$, $\forall_j y \in Y_j$ and $\sum_{i=1}^n x_i = \sum_{i=1}^n e_i + \sum_{j=1}^J y_j$.

Definition (Competitive equilibrium) CE $= \left\{ p^\star, (x_i^\star)_{i=1}^n, (y_i^\star)_{j=1}^J \right\} \in \overline{\Delta} \times \mathbb{R}_+^{ln} \times \mathbb{R}^{lJ}$ is a competitive equilibrium in $\mathcal{E}$ if $\forall_i x_i^\star \in x_i(p^\star)$, $\forall_j y_j^\star \in \underset{y \in Y_j}{\operatorname{argmax}} p \cdot y$ and $\sum_{i=1}^n x_i^\star = \sum_{i=1}^n e_i + \sum_{j=1}^J y_j^\star$. That is, $x_i^\star$ is in the demand correspondence of each agent, each firm is maximizing profits given p, and all markets clear. (clearly $\pi_j^\star = p^\star \cdot y_j^\star$).

Note: For many applications a quasi-equilibrium is defined where agents are require to minimize expenditure instead of maximize utility. A quasi-equilibrium is also a CE if consumption sets are convex and preferences continuous.

Definition (Weak Pareto optimum) A *feasible* allocation $(x,y) \in \mathbb{R}_+^{ln} \times \mathbb{R}^{lJ}$ is weak Pareto optimal if there is no other *feasible* allocation $(x', y') \in \mathbb{R}_+^{ln} \times \mathbb{R}^{lJ}$ that strongly dominates (x,y) $\left(\forall_i x_i' \succ_i x_i' \right)$.

Definition (Strong Pareto optimum) A *feasible* allocation $(x, y) \in \mathbb{R}_+^{ln} \times \mathbb{R}^{lJ}$ is strong Pareto optimal if there is no other *feasible* allocation $\left(x^{'}, y^{'}\right) \in \mathbb{R}_+^{ln} \times \mathbb{R}^{lJ}$ that strongly dominates (x, y) $\left(\forall_i x_i^{'} \succeq_i x_i, \exists_k x_k^{'} \succ_k x_k\right)$.

Note: The concepts of weak and strong Pareto dominance do not change between the pure exchange and the production economy. Then Pareto optimality only changes because the concept of feasibility changes. The value of y in an allocation is only relevant for Pareto optimality as it affects what is considered feasible.

Theorem (Existence of competitive equilibrium) Let X_i be closed and convex, $\succeq_i$ a complete preorder, continuous, l.n.s and convex for all i, and Y_j closed, convex and such that includes the origin and satisfies free disposal for all j. Then a CE exists.

Theorem (First welfare theorem) Let $\mathcal{E} = \left\{\left(\succeq_i, e_i, \{\theta_{ij}\}_{j=1}^J\right)_{i=1}^n, \{Y_j\}_{j=1}^J\right\}$ be a production economy with $\succeq_i$ a continuous complete preorder and $e_i \in \mathbb{R}_+^l$ for all i, moreover $\sum e_i \gg 0$. Then, if $\succeq_i$ are locally non-satiated and $\left\{p^\star, (x_i^\star)_{i=1}^n, (y_i^\star)_{j=1}^J\right\} \in \Delta \times \mathbb{R}_+^{ln} \times \mathbb{R}^{lJ}$ is a competitive equilibrium, $x^\star$ is a (strong) Pareto optimum.

Proof:

- Let $\left\{p^\star, (x_i^\star)_{i=1}^n, (y_i^\star)_{j=1}^J\right\}$ be a competitive equilibrium for $\mathcal{E}$. Define the wealth of each trader i as $w_i = p^\star \cdot e_i + \sum\limits_{j=1}^J \theta_{ij}\pi_j$.

- Suppose for contradiction that $\left\{(x_i^\star)_{i=1}^n, (y_i^\star)_{j=1}^J\right\}$ is not Pareto optimum. Then there exists $\left((x_i)_{i=1}^n, (y_i)_{j=1}^J\right) \in \mathbb{R}_+^{ln} \times \mathbb{R}^{lJ}$ such that is feasible $\left(\sum\limits_{i=1}^n x_i = \sum\limits_{i=1}^n e_i + \sum\limits_{j=1}^J y_j\right)$ and $\forall_i x_i \succeq_i x_i^\star, \exists_k x_k \succ_k x_k^\star$.

- $\forall_i p^\star \cdot x_i \geq w_i$: Suppose not, then $\exists_i p^\star \cdot x_i < w_i$, by l.n.s. there exits $x_i^{'}$ such that $x_i^{'} \succ_i x_i$ and $p^\star \cdot x_i^{'} < w_i$. That is, $x^{'}$ is affordable at prices $p^\star$ for trader i. By transitivity $x_i^{'} \succ x_i \succeq x_i^\star$ which contradicts $x_i^\star$ being optimal for trader i.

- $p^\star \cdot x_k > w_k$: Suppose not, then x_k would be affordable for trader k, since $x_k \succ_k x_k^\star$ this contradicts $x_k^\star$ being optimal for trader k.

- Since firms are profit maximizing it must be that $\forall_j \pi_j = p^\star \cdot y_j^\star \geq p^\star \cdot y_j$.

- Then
$$\sum_i^n p^\star \cdot x_i > \sum_i^n w_i = \sum_i^n p^\star \cdot e_i + \sum_{j=1}^J \pi_j \geq \sum_i^n p^\star \cdot e_i + \sum_{j=1}^J p^\star \cdot y_j$$

which implies: $p^{\star} \cdot \left[\sum_{i}^{n}(x_i - e_i) - \sum_{j=1}^{J} y_j\right] > 0$. Since $p^{\star} \in \overline{\Delta}$ this implies $\exists_m \sum_{i}^{n}(x_i^m - e_i^m) - \sum_{j=1}^{J} y_j^m > 0$ which contradicts feasibility of (x, y).

Theorem (Second welfare theorem) Let $(x, y) \in \mathbb{R}_{+}^{ln} \times \mathbb{R}^{lJ}$ be a (strong) Pareto optimum for a production economy $\mathcal{E} = \left\{\left(\succeq_i, e_i, \{\theta_{ij}\}_{j=1}^{J}\right)_{i=1}^{n}, \{Y_j\}_{j=1}^{J}\right\}$ with $\succeq_i$ continuous, strictly monotone and strictly convex, and Y_j closed, convex and nonempty. Then there exists price vector p and a reallocation of endowments and firm shares $\left(e_i^{'}, \{\theta_{ij}^{'}\}_{j=1}^{J}\right)_{i=1}^{n}$ such that $\{p, x, y\}$ is a competitive equilibrium for $\mathcal{E}^{'} = \left\{\left(\succeq_i, e_i^{'}, \{\theta_{ij}^{'}\}_{j=1}^{J}\right)_{i=1}^{n}, \{Y_j\}_{j=1}^{J}\right\}$. Moreover, (x, y) is the only competitive equilibrium allocation and the price vector p satisfies $p \geq 0$ and $p \neq 0$.

Note: Strict monotonicity can be weakened to local non-satiation. Strict convexity is only needed for $\overline{x}$ to be the only equilibrium allocation, it can be weakened to convexity.

Part IV

Aldo Rustichini

20 Game Form and Preferences over Consequences

Definition (Game Form) A game form is a tuple $G = \left\{I, \{A^i\}_{i \in I}, C, g\right\}$ formed by:

i. A finite set of players $I = \{1, \ldots, n\}$.

ii. A finite set of actions for every player $A^i = \left\{a_1^i, \ldots, a_{k^i}^i\right\}$, where k^i is the number of actions of player i.

iii. A finite set of consequences $C = \{c_1, \ldots, c_m\}$.

iv. A function $g : A \to C$ that assigns consequences to action profiles $a \in A = \underset{i \in I}{\times} A^i$ where $\times$ indicates cartesian product.

Definition (Simple Lottery) Let C be the (finite) set of consequences. A simple lottery $L \in \Delta_m$ is a probability distribution over the space of consequences. $L = \begin{pmatrix} p_1 & \cdots & p_m \\ c_1 & \cdots & c_m \end{pmatrix}$ with $p_j \geq 0$ and $\sum_{j=1}^{m} p_j = 1$. Lottery L assigns probability p_j to outcome c_j.

Space of Simple Lotteries The space of simple lotteries is noted $\mathcal{G}^0 = \Delta_m$. These are gambles of order 0.

Definition (Compound Lottery) Let $k \in \mathbb{N}$ and $\{L_1, \ldots, L_k\} \subset \mathcal{G}^0$. A compound lottery $M = \begin{pmatrix} q_1 & \cdots & q_k \\ L_1 & \cdots & L_k \end{pmatrix}$ is a probability distribution over a subset of simple lotteries. Compound lottery M gives lottery L_j with probability q_j. These are gambles of order 1, $\mathcal{G}^1$.

Arbitrary Compound Lotteries An n^{th} order compound lottery $M \in \mathcal{G}^n$ is an element of the form $M = \begin{pmatrix} q_1 & \cdots & q_k \\ M_1 & \cdots & M_k \end{pmatrix}$ where $M_j \in \mathcal{G}^{n-1}$.

Definition (Space of Lotteries) Let $\mathcal{G} = \bigcup_{n=0}^{\infty} \mathcal{G}^n$ be the set of all lotteries.

Definition (Reduction) Let $M \in \mathcal{G} \setminus \mathcal{G}^0$. $R(M) \in \mathcal{G}^0$ is the reduction of M to a simple lottery. Denote by $R(M)(c_j)$ the probability given by $R(M)$ to consequence c_j.

For $M \in \mathcal{G}^1$ the probability distribution that defines $\tilde{R}(M)$ is given by

$$\tilde{R}(M)(c_j) = \sum_{i=1}^{k} q_i p_j^i$$

where μ_j^i is the probability assigned to c_j by simple lottery L_i. Note that $\tilde{R}(M)(c_j) \geq 0$ since $q_i, p_j^i \geq 0$ for all i, j and thus $\tilde{R}(M) \in \mathcal{G}^0$:

$$\sum_{j=1}^{m} \tilde{R}(M)(c_j) = \sum_{j=1}^{m}\sum_{i=1}^{k} q_i p_j^i = \sum_{i=1}^{k} q_i \left(\sum_{j=1}^{m} p_j^i \right) = \sum_{i=1}^{k} q_i = 1$$

The reduction of an n order gamble M to an $n-1$ order gamble is noted as $\tilde{R}^n(M) \in \mathcal{G}^{n-1}$, for $n \geq 2$. M has the form $M = \begin{pmatrix} p_1 & \cdots & p_J \\ L_1 & \cdots & L_J \end{pmatrix}$, where $p_j \geq 0$, $\sum p_j = 1$ and $L_j \in \mathcal{G}^{n-1}$. Let $G_1, \ldots G_K \in \mathcal{G}^{n-2}$ be all the lotteries that form the possible outcomes of $L_1, \ldots, L_J$. Define $R^n(M) = \begin{pmatrix} q_1 & \cdots & q_K \\ G_1 & \cdots & G_K \end{pmatrix}$ where $q_k = \sum_{j=1}^{J} p_j s_k^{L_j}$ and $s_k^{L_j}$ is the probability assigned by gamble L_j to outcome G_k, if G_k is not an outcome of L_j then $s_k^{L_j} = 0$. Note that since G_k is an outcome for at least one gamble L_j and $p_j \geq 0$ for all j it follows that $q_k \geq 0$ for all k. Also, $\sum_{k=1}^{K} q_k = \sum_{k=1}^{K}\sum_{j=1}^{J} p_j r_k^{L_j} = \sum_{j=1}^{J} p_j \sum_{k=1}^{K} r_k^{L_j} = \sum_{j=1}^{J} p_j = 1$. Then $R^n(M) \in \mathcal{G}^{n-1}$.

Finally the reduction of an n order gamble M to a simple gamble is defined as $R(M) = \tilde{R} \circ \tilde{R}^2 \circ \cdots \circ \tilde{R}^n(M)$.

Definition (Preference Relation) $\succeq$ is a binary relation defined over $\mathcal{G}$.

Definition (Utility Representation) The function $u : \mathcal{G} \to \mathbb{R}$ represents $\succeq$ if for all $L, M \in \mathcal{G}$: $L \succeq M \iff u(L) \geq u(M)$.

Expected Utility Property (EUP) Function u has the EUP if there exists $v : C \to \mathbb{R}$ such that for $L \in \mathcal{G}^0$: $u(L) = \sum_{j=1}^{m} p_j v(c_j)$.

Properties of Preference Relation

i. Weak Order: $\succeq$ is a complete and transitive relation.

ii. Continuity: For all $L \in \mathcal{G}$ there exists $\alpha \in [0,1]$ such that: $M \sim c_1 \alpha c_m = \begin{pmatrix} \alpha & \cdots & 1-\alpha \\ c_1 & \cdots & c_m \end{pmatrix}$

iii. Monotonicity: For $\alpha, \beta \in [0,1]$: $\alpha \geq \beta \iff c_1 \alpha c_m \succeq c_1 \beta c_m$.

iv. Substitution: Consider $L = \begin{pmatrix} q_1 & \cdots & q_K \\ L_1 & \cdots & L_K \end{pmatrix}$ and $M = \begin{pmatrix} q_1 & \cdots & q_K \\ M_1 & \cdots & M_K \end{pmatrix}$. If $\forall_k L_k \sim M_k \to L \sim M$.

v. Reduction: For all $M \in \mathcal{G} \setminus \mathcal{G}^0$: $M \sim R(M)$.

Order of Consequences: Under the Weak Order property there must be a best and worst outcome. WLOG $c_1 \succeq \cdots \succeq c_m$.

Theorem (VNM - Expected Utility Representation)

i. Existence: Preference relation $\succeq$ has a utility representation with the EUP if and only if it satisfies the five properties above.

ii. Uniqueness: The representation is unique up to monotonic affine transformations.

(a) If u represents $\succeq$ and has the EUP then for $A > 0$ and $B \in \mathbb{R}$ $v = Au + B$ also represents $\succeq$ and has the EUP.

(b) If u and v represent $\succeq$ and have the EUP then there exists $A > 0$ and $B \in \mathbb{R}$ such that $v = Au + B$.

Note: Due to the reduction property the representation we have for all $M \in \mathcal{G}$ that $u(M) = \sum R(M)(c_j) v(c_j)$. All results can be then established considering only simple lotteries with the payoffs given by the expected utility representation.

21 Normal Form Games

21.1 Nash Equilibria

Definition (Normal Form Game) A normal form game G induced by game form $G' = \left\{I, \{A^i\}_{i\in I}, C, g\right\}$ and preferences $\{\succeq^i\}$ over $\mathcal{G}$ with expected utility representation $\{w^i\}$, is a tuple $G = \left\{I, \{A^i\}_{i\in I}, \{u^i\}_{i\in I}\right\}$ formed by:

i. A finite set of players $I = \{1, \ldots, n\}$.

ii. A finite set of actions for every player $A^i = \left\{a^i_1, \ldots, a^i_{k^i}\right\}$, where k^i is the number of actions of player i.

iii. A payoff (utility) function $u^i : A \to \mathbb{R}$ where $A = \times A^i$.

Definition (Pure Strategy) A pure strategy is an object of the form $a = (a^1, \ldots, a^n) \in A$.

Definition (Mixed Strategy) A mixed strategy for player i is an object of the form $s^i \in S^i = \Delta(A^i)$, that is, a probability distribution over the player's actions. A mixed strategy profile is then $s \in S = \underset{i\in I}{\times} S^i$.

Degenerated mixed strategies Let $a \in A^i$. Denote by $s^i_a \in S^i$ the degenerated mixed strategy over a. $s^i_a(a) = 1$ and $s^i_a(b) = 0$ for $b \in A^i \setminus \{a\}$.

Probability over action profiles Note that a mixed strategy s induces a probability over the space A of action profiles: $\Pr_s(a) = \prod_{i=1}^{n} s^i(a^i)$.

Utility under mixed strategies The (expected) utility of playing action $a \in A^i$ for player i given that other players are playing according to mixed strategy s^{-i} is given by:

$$\begin{aligned}
\forall_{a\in A^i} \quad u^i(a, s^{-i}) &= \sum_{a^{-i}\in A^{-i}} \left(\prod_{j\neq i} s_j(a^j)\right) u\left(a, a^{-i}\right) = \sum_{a^{-i}\in A^{-i}} \Pr_{s^{-i}}\left(a^{-i}\right) u\left(a, a^{-i}\right) \\
\forall_{t\in S^i} \quad u^i(t, s^{-i}) &= \sum_{a\in A^i} t(a)\, u^i\left(a, s^{-i}\right)
\end{aligned}$$

Definition (Best Response - Pure Strategies) A best response to other players playing strategy profile $s^{-i} \in S^{-i}$ is a correspondence $\mathrm{BR}^i_{A^i} : S \rightrightarrows A^i$

$$\mathrm{BR}^i_{A^i}\left(s^i, s^{-i}\right) = \mathrm{BR}^i_{A^i}(s) = \left\{b \in A^i | \forall_{c\in A^i} u^i\left(b, s^{-i}\right) \geq u^i\left(c, s^{-i}\right)\right\} = \underset{b\in A^i}{\operatorname{argmax}}\, u^i\left(b, s^{-i}\right)$$

Note that the best responds does not actually depend on the player's own action. It is included as an argument for convenience. Define $\mathrm{BR}_A(s) = \underset{i\in I}{\times} \mathrm{BR}^i_{A^i}(s)$.

Definition (Best Response - Mixed Strategies) A best response to other players playing mixed strategy profile $s^{-i} \in S^{-i}$ is a correspondence $\mathrm{BR}^i_{S^i} : S \rightrightarrows S^i$

$$\mathrm{BR}^i_{S^i}\left(s^i, s^{-i}\right) = \mathrm{BR}^i_{S^i}\left(s\right) = \left\{t \in S^i | \forall_{r \in S^i} u^i\left(t, s^{-i}\right) \geq u^i\left(r, s^{-i}\right)\right\} = \underset{t \in S^i}{\operatorname{argmax}}\, u^i\left(t, s^{-i}\right)$$

Note that the best responds does not actually depend on the player's own action. It is included as an argument for convenience. Define $\mathrm{BR}_S\left(s\right) = \underset{i \in I}{\times} \mathrm{BR}^i_{S^i}\left(s\right)$.

Definition (Nash Equilibria) A NE is a mixed strategy profile $s \in S$ such that no player will benefit from unilaterally deviating from the strategy. That is $s \in S$ is a NE if $s \in \mathrm{BR}_S\left(s\right)$, so that s^i is a best response for s^{-i} for all agents. This is a fixed point of BR_S.

A NE in pure strategies is a fixed point of BR_A, the best responses in pure strategies.

Definition (Nash Equilibria Payoffs) $\mathrm{NEP} = \left\{x \in \mathbb{R}^n | x^i = \sum_{a \in A} u^i\left(a\right) \mathrm{Pr}_s\left(a\right) \wedge s \in \mathrm{NE}\right\}$.

Proposition (Properties of BR)

i. For every i the correspondence $\mathrm{BR}^i_{A^i}$ is non-empty valued, finite valued and upper hemi-continuous.

ii. For every i it holds that: $\mathrm{BR}^i_{S^i}\left(s\right) = \mathrm{co}\left(\left\{s_a \in S^i | a \in \mathrm{BR}^i_{A^i}\left(s\right)\right\}\right)$,where $\mathrm{co}\left(\cdot\right)$ is the convex hull and s_a is a degenerate mixed strategy.

iii. The correspondence $\mathrm{BR}^i_{S^i}$ is non-empty valued, convex valued, compact valued (since $\mathrm{BR}^i_{A^i}$ is finite valued) and upper hemi-continuous.

iv. The correspondence BR_S is also non-empty valued, convex valued, compact valued (since $\mathrm{BR}^i_{A^i}$ is finite valued) and upper hemi-continuous.

Theorem (Existence of NE in Mixed Strategies) There exists a NE in mixed strategies. The result is obtained by Kakutani's fixed point theorem, applied to the correspondence BR_S.

Proposition The set of NE is closed.

Proof:

- Let $\{s_n\} \subset NE$ and $s_n \rightarrow s$.
- Since $s_n \in NE$ for all n $s_n \in \mathrm{BR}_S\left(s_n\right)$, then s_n is in the "image" of s_n, by u.h.c the sequence $\{s_n\}$ has a convergent subsequence such that $s_{n_k} \rightarrow s^{'} \in \mathrm{BR}_S\left(s\right)$.
- Since $s_n \rightarrow s$ one gets $s^{'} = s$ which is $s \in \mathrm{BR}_S\left(s\right)$. Then $s \in NE$ which proves closedness of the set.

21.2 Perfect Equilibria

Definition (Perturbation) A perturbation is$\eta \in \underset{i \in I}{\times} \mathbb{R}^{k^i}_{++}$, where $k^i = |A^i|$. $\eta = (\eta^1, \ldots, \eta^n)$.

Definition (Perturbed Strategy) The set of perturbed strategies profiles is $S_\eta = \times S^i_{\eta^i}$, where $S^i_{\eta^i} = \{s^i \in S^i | s^i \geq \eta^i > 0\}$. A perturbed strategy is $s^i_{\eta^i} \geq \eta^i$.

Note: These sets are still convex and compact, and non-empty given η.

Definition (η-Equilibria) A perturbed strategy $\hat{s}(\eta)$ is an η-equilibrium if for all i:

$$\hat{s}(\eta) \in \mathrm{BR}^i_{S^i_{\eta^i}}(\hat{s}(\eta)) = \underset{r \in S^i_{\eta^i}}{\operatorname{argmax}}\ u^i\left(r, \hat{s}^{-i}(\eta)\right)$$

Definition (Perfect Equilibria) A mixed strategy $s \in S$ is a perfect equilibrium if there exists a sequence $\{\eta_n\}$ of perturbations such that $\eta_n \to 0$ and a sequence of s_n of mixed strategies such that for all n s_n is an η_n-equilibrium and $s_n \to s$.

Theorem (Existence of η-Equilibria) Note that for all η S_η is still compact and convex, using ToM the constraint best responses are still non-empty, convex, compact valued and u.h.c. Then by Kakutani's fix point theorem an η-equilibrium exists.

Theorem (Existence of Perfect Equilibria) Note that for all η_n in the sequence there exists a s_n that is an η_n-equilibrium. Also $\{s_n\} \subset S$ which a compact set, then the sequence $\{s_n\}$ has a convergent subsequence. The limit of the subsequence is a perfect equilibrium.

Proposition Every Perfect Equilibrium is a Nash Equilibrium

Proof: Let s be a perfect equilibria. Then there exists $\eta_n \to 0$ and $s_n \to s$ such that s_n is an η_n-equilibria of the perturbed game. Then for each n and each i:

$$\sum_{a \in A} \left(\prod_j s^j_n \right) u^i(a) \geq \sum_{a \in A} t^i_\eta \left(\prod_{j \neq i} s^j_n \right) u^i(a)$$

for all $t^i_\eta \in S^i$ such that $t^i_\eta \geq \eta^i_n$.

Let $t^i \in S^i$ taken arbitrarily. Since $\eta^i_n \to 0$ there exists a sequence $\{t^i_n\}$ such that $t^i_n \geq \eta^i_n$ and $t^i_n \to t^i$. From above it follows that:

$$\sum_{a \in A} \left(\prod_j s^j_n \right) u^i(a) \geq \sum_{a \in A} t^i_n \left(\prod_{j \neq i} s^j_n \right) u^i(a)$$

for all n. Taking limits:

$$\sum_{a \in A} \left(\prod_j s^j \right) u^i(a) \geq \sum_{a \in A} t^i \left(\prod_{j \neq i} s^j \right) u^i(a)$$

Since t^i wash chosen arbitrarily the above holds for all $t^i \in S^i$. Then s^i is a best response to s^{-i} for all players, hence s is a NE of the game.

Proposition A perfect equilibrium assigns zero probability to (pure) strategies that are weakly dominated.

Proposition In a two player game a NE is a Perfect equilibrium if and only if it has no strategy that is weakly dominated.

Proposition The set of perfect equilibria is closed.

Proof: Let $\{s_n\}$ be a sequence of perfect equilibria such that $s_n \to s \in S$ (a mixed strategy).

- For arbitrary $m \in \mathbb{N}$ there exists N_m such that $\|s - s_{N,m}\| \leq \frac{1}{2m}$.
- Since $s_{N,m}$ is a perfect equilibrium there exists k such that $\|s_n - s_{N_k,m}\| < \frac{1}{2m}$ where $s_{N_k,m}$ is a $\eta_{N_k,m}$-equilibria. Then $\|s - s_{N_k,m}\| \leq \|s - s_{N,m}\| + \|s_{N,m} - s_{N_k,m}\| = \frac{1}{m}$.
- For each m rename $\eta_{N_k,m}$ as η_m and $s_{N_k,m}$ as s_m. It is clear that s_m is an η_m-equilibria for all m and that $s_m \to s$.
- It must be that s is a perfect equilibrium. (it can be that $\eta_m \to 0$ or not, if it converges to zero s is automatically a perfect equilibria, if not then it must be that s is fully mixed since $s_m \to s$ and s_m is a perturbed strategy, hence s is a perturbed equilibrium itself. In this case s is a perfect equilibria since it is the limit of a constant sequence of perturbed equilibria).

21.2.1 Proper Equilibria

Definition (η-Proper) Let G be a NFG and $\eta > 0$. A fully mixed strategy profile is η-Proper if:

$$\forall_i \forall_{a,b \in A^i} \left[u^i\left(a, s^{-i}\right) < u^i\left(b, s^{-i}\right) \longrightarrow s^i(a) \leq \eta s^i(b) \right]$$

Definition (Proper Equilibria) A mixed strategy profile s is a proper equilibrium if there exists a sequence $\eta_n \to 0$ and a sequence $s_n \to s$ such that s_n is η_n-Proper.

Theorem (Myerson) Every finite forma game has a proper equilibrium.

21.3 Correlated Equilibria

Definition (Correlated Strategy) A correlated strategy μ is a probability over action profiles. $\mu \in \Delta(A)$.

Note: The probability over action profiles induced by a mixed strategy s, $\Pr_s$, is a correlated strategy.

Definition (Marginal Probability) Let $\mu \in \Delta(A)$ be a correlated strategy. μ_{A^i} is the marginal distribution of μ over actions of player i. For $a \in A^i$: $\mu_{A^i}(a) = \sum\limits_{a^{-i} \in A^{-i}} \mu(a, a^{-i})$.

Note: $\mu_{A^i}(a) = 0$ if an only if $\forall_{a^{-i} \in A^{-i}} \mu(a, a^{-i}) = 0$.

Definition (Conditional Probability) Let $\mu \in \Delta(A)$ be a correlated strategy. $\mu(\cdot|b) \in \Delta(A^{-i})$ is the probability of $a^{-i} \in A^{-i}$ given μ and $b \in A^i$. It is defined as: $\mu(a^{-i}|b) = \frac{\mu\left(b, a^{-i}\right)}{\mu_{A^i}(b)}$.

Definition (Correlated Equilibria) A correlated strategy $\mu \in \Delta(A)$ is a correlated equilibrium if;

$$\forall_i \forall_{a \in A^i} \left[\mu_{A^i}(a) > 0 \to \forall_{c \in A^i} \sum_{a^{-i} \in A^{-i}} u^i\left(a, a^{-i}\right) \mu\left(a^{-i}|a\right) \geq \sum_{a^{-i} \in A^{-i}} u^i\left(c, a^{-i}\right) \mu\left(a^{-i}|a\right) \right]$$

or equivalently:

$$\forall_i \forall_{a \in A^i} \forall_{c \in A^i} \sum_{a^{-i} \in A^{-i}} \left(u^i\left(a, a^{-i}\right) - u^i\left(c, a^{-i}\right)\right) \mu\left(a, a^{-i}\right) \geq 0$$

Since $\mu_{A^i}(a) = 0 \iff \forall_{a^{-i} \in A^{-i}} \mu(a, a^{-i}) = 0$ then the condition is satisfied with equality when $\mu_{A^i}(a) = 0$.

Definition (Correlated Equilibria Payoffs) $\text{CEP} = \left\{ x \in \mathbb{R}^n | x^i = \sum\limits_{a \in A} u^i(a) \mu(a) \wedge \mu \in \Delta(A) \right\}$.

Theorem (Existence of Correlated Equilibria) The set of CE is non-empty since for s a NE the probability over actions profile $\Pr_s$ is a CE.

Proof:

- Let s be a NE and $\Pr_s(a) = \prod\limits_{i=1}^{n} s^i(a^i)$ its induced probability over action profiles. Note from the definition of the NE that

$$s^i \in \underset{t \in S^i}{\operatorname{argmax}} \left[u\left(t, s^{-i}\right)\right] = \underset{t \in S^i}{\operatorname{argmax}} \left[\sum_{a_i \in A^i} t(a_i) \sum_{a^{-i} \in A^{-i}} \prod_{j \neq i} s_j(a_j) u\left(a_i, a^{-i}\right) \right]$$

This implies that $s^i(a_i) > 0$ if and only if, for all $c \in A^i$:

$$\sum_{a^{-i}\in A^{-i}} \prod_{j\neq i} s_j(a_j)\, u\left(a_i, a^{-i}\right) \geq \sum_{a^{-i}\in A^{-i}} \prod_{j\neq i} s_j(a_j)\, u\left(c, a^{-i}\right)$$

- Note that since $\mu = \Pr_s$ we have $\mu(a) = \prod s_i(a_i)$, and then $\mu_{A^i}(a_i) = s^i(a_i) \sum\limits_{a^{-i}\in A^{-i}} \prod\limits_{j\neq 1} s_j(a_j) = s^i(a_i)$ and $\mu(a^{-i}|a_i) = \prod\limits_{j\neq 1} s_j(a_j)$.

- Then the condition above gives:

$$\sum_{a^{-i}\in A^{-i}} \mu\left(a^{-i}|a_i\right) u\left(a_i, a^{-i}\right) \geq \sum_{a^{-i}\in A^{-i}} \mu\left(a^{-i}|a_i\right) u\left(c, a^{-i}\right)$$

  for all player and all action a_i such that $\mu_{A^i}(a_i) = s^i(a_i) > 0$. Then $\mu = \Pr_s$ is a Correlated Equilibrium.

Proposition

i. The set of CE is convex and compact since is defined by a finite set of linear equations.

ii. The set of CEP is convex and compact.

iii. $\mathrm{co}\,(\mathrm{NEP}) \subseteq \mathrm{CEP}$ with strict inclusion for some games.

21.4 Min-Max Theorem

Definition (Zero Sum Game) A two players finite action normal form game is zero sum if the sum of the utilities of the two players is equal to 0 for any action profile. Let $u_1 = u$,so $u_2 = -u$.

Definition (MinMax Theorem) For a zero sum game of two players:

$$\min_{s^2\in\Delta(A^2)} \max_{s^1\in\Delta(A^1)} u\left(s^1, s^2\right) = \max_{s^1\in\Delta(A^1)} \min_{s^2\in\Delta(A^2)} u\left(s^1, s^2\right)$$

Proof:

i. Note that for any $\bar{s}^1 \in \Delta\left(A^1\right)$ and $\bar{s}^2 \in \Delta\left(A^2\right)$ it holds that:

$$u\left(\bar{s}^1, \bar{s}^2\right) \geq \min_{s^2\in\Delta(A^2)} u\left(\bar{s}^1, s^2\right)$$

Then by taking maximum at both sides with respect to s^1:

$$\max_{s^1\in\Delta(A^1)} u\left(s^1, \bar{s}^2\right) \geq \max_{s^1\in\Delta(A^1)} \min_{s^2\in\Delta(A^2)} u\left(s^1, s^2\right)$$

Note that the RHS is now constant, and a lower bound to the LHS across s^2, then:

$$\min_{s^2\in\Delta(A^2)} \max_{s^1\in\Delta(A^1)} u\left(s^1, s^2\right) \geq \max_{s^1\in\Delta(A^1)} \min_{s^2\in\Delta(A^2)} u\left(s^1, s^2\right)$$

ii. Note that for any $\bar{s}^1 \in \Delta\left(A^1\right)$ it holds that:

$$\max_{s^1\in\Delta(A^1)} \min_{s^2\in\Delta(A^2)} u\left(s^1, s^2\right) \geq \min_{s^2\in\Delta(A^2)} u\left(\bar{s}^1, s^2\right)$$

In particular for $\hat{s}^1$ a NE of the game the inequality must hold. Note that if $(\hat{s}^1, \hat{s}^2)$ it is defined as an strategy profile such that:

$$u\left(\hat{s}^1, \hat{s}^2\right) = \max_{s^1\in\Delta(A^1)} u\left(s^1, \hat{s}^2\right) \qquad -u\left(\hat{s}^1, \hat{s}^2\right) = \max_{s^2\in\Delta(A^2)} -u\left(\hat{s}^1, s^2\right)$$

The second condition implies:

$$u\left(\hat{s}^1, \hat{s}^2\right) = \min_{s^2\in\Delta(A^2)} u\left(\hat{s}^1, s^2\right) = \max_{s^1\in\Delta(A^1)} u\left(s^1, \hat{s}^2\right)$$

Then the initial condition gives (evaluated at $\hat{s}^1$):

$$\begin{aligned}
\max_{s^1\in\Delta(A^1)} \min_{s^2\in\Delta(A^2)} u\left(s^1, s^2\right) &\geq \min_{s^2\in\Delta(A^2)} u\left(\hat{s}^1, s^2\right)\\
\max_{s^1\in\Delta(A^1)} \min_{s^2\in\Delta(A^2)} u\left(s^1, s^2\right) &\geq \max_{s^1\in\Delta(A^1)} u\left(s^1, \hat{s}^2\right) \geq \min_{s^2\in\Delta(A^2)} \max_{s^1\in\Delta(A^1)} u\left(s^1, s^2\right)
\end{aligned}$$

Where the second inequality follows by the definition of minimum. The first and last term of the inequalities give the result.

Definition (Value of a Game) For a zero sum game of two players define the value of the game as $V:\mathbb{R}^{nm}\to\mathbb{R}$ (where $n=\#A^1$ and $m=\#A^2$):

$$V(u)=\max_{s^1\in\Delta(A^1)}\min_{s^2\in\Delta(A^2)}U\left(s^1,s^2|u\right)$$

For a given strategy profile $s^1=(p_1,\ldots,p_n)$, $s^2=(q_1,\ldots,q_n)$ and payoffs $u\in\mathbb{R}^{nm}$ we define $U\left(s^1,s^2|u\right)=\sum\limits_{i=1}^{n}\sum\limits_{j=1}^{m}p_iq_ju_{ij}$.

Proposition (Properties of the Value of a Game) The value of a game is continuous, non-decreasing and homogenous of degree one in payoffs.

Proof:

i. Consider the problem:

$$v\left(s^1,u\right)=\min_{s^2\in\Delta(A^2)}U\left(s^1,s^2|u\right)$$

note that U is continuous in s_1,s_2 and u and that the minimum is being taken over s^2 in a compact set that does not vary with s^1 or u. By the theorem of the maximum the value of this problem, as a function of s^1 and u is a continuous function. Now consider:

$$V(u)=\max_{s^1\in\Delta(A^1)}\min_{s^2\in\Delta(A^2)}U\left(s^1,s^2|u\right)=\max_{s^1\in\Delta(A^1)}v\left(s^1,u\right)$$

again since v is continuous and s^1 varies in a compact set independent of u by the theorem of the maximum V is a continuous function of u.

ii. Let $u_1\leq u_2$. Clearly for all s^1,s^2: $U\left(s^1,s^2|u_1\right)\leq U\left(s^1,s^2|u_2\right)$. Then:

$$\min_{s^2\in\Delta(A^2)}U\left(s^1,s^2|u_1\right)\leq\min_{s^2\in\Delta(A^2)}U\left(s^1,s^2|u_2\right)$$

$$V(u_1)=\max_{s^1\in\Delta(A^1)}\min_{s^2\in\Delta(A^2)}U\left(s^1,s^2|u_1\right)\leq\max_{s^1\in\Delta(A^1)}\min_{s^2\in\Delta(A^2)}U\left(s^1,s^2|u_2\right)=V(u_2)$$

iii. Let $\lambda\in\mathbb{R}$, note that $U\left(s^1,s^2|\lambda u\right)=\lambda U\left(s^1,s^2|u\right)$ and $\max\lambda f(x)=\lambda\max f(x)$.

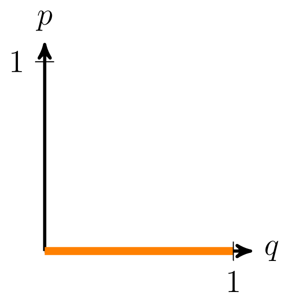
21.5 Best Response Functions in 2x2 NFG

Consider the general 2×2 game with payoffs for player 1 listed below

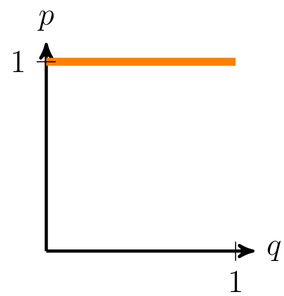
	L	R
T	a	b
B	c	d

In the 2×2 the following best response functions are possible for player 1. The vertical axis represents the probability player 1 attaches to action T in response to player 2's probability q of playing action L.

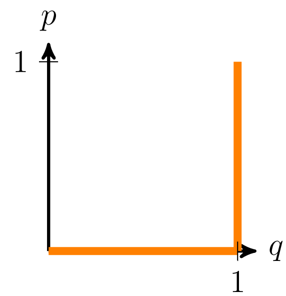
There are only 9 possible types of best response functions that depend in how the player 1's strategies are compared, given an strategy of player 2. The relevant comparisons are then between a and c and between b and d. p is the probability of player 1 playing T and q is the probability of player 2 playing L.



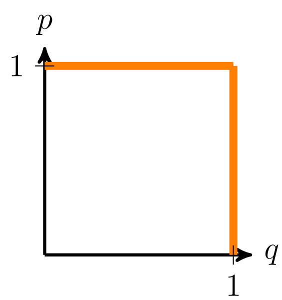
(a) $a < c, b < d$



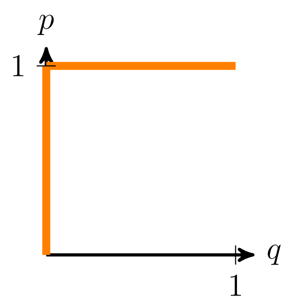
(b) $a > c, b > d$



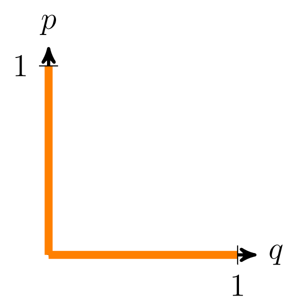
(c) $a = c, b < d$



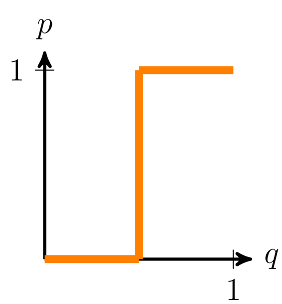
(d) $a = c, b > d$



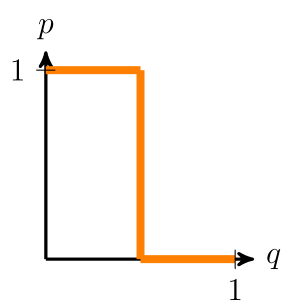
(e) $a > c, b = d$



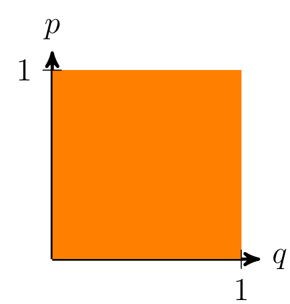
(f) $a < c, b = d$



(g) $a > c, b < d$



(h) $a < c, b > d$



(i) $a = c, b = d$

22 Extensive Form Games

Definition (Extensive Form Game) An Extensive form game is a tuple

$$G = \left\{ I, \alpha, p, X, Z, \succeq, P, \left\{ V^i \right\}_{i \in I}, \left\{ u^i \right\}_{i \in I}, C \right\}$$

formed by:

i. A finite set of players $I = \{1, \dots, n\}$.

ii. An initial node α where "nature" moves.

iii. A probability distribution P over "nature's" actions, C_α.

iv. A set of move nodes for players X.

v. A set of final nodes Z.

vi. A binary relation $\succeq$ defined over the nodes $N = \alpha \cup X \cup Z$.

   (a) $x \succeq y$ is read x comes after y.
   (b) For all $z \in Z$ there exists $x \in X$ such that $z \succeq x$.
   (c) For all $x \in X$ the set $\{y \in N | x \succeq y \succeq \alpha\}$ is linearly ordered. $\succeq$ is complete and transitive on the set.

vii. A partition of the move nodes into the move nodes of each player $P = \{P^1, \dots, P^n\}$ with $X = \bigcup\limits_{i \in I} P^i$ and $P^i \bigcap P^j = \emptyset$ for $i \neq j$.

viii. Information partitions for each player's move nodes $V^i = \left\{ v_1^i, \dots, v_{k^i}^i \right\}$ with $P^i = \bigcup\limits_{j=1}^{k^i} v_j^i$ and $v_m^i \bigcap v_n^i = \emptyset$ for $m \neq n$.

ix. A payoff (utility) function defined over the set of final nodes (Z) given by: $u^i(z)$.

x. A correspondence C that assigns actions to information sets. So that C_v are the actions of player i at move nodes in information set v.

EFG A game is usually only represented by a tuple $G = \left\{ X, Z, P, \{V^i\}_{i \in I}, \{u^i\}_{i \in I} \right\}$ omitting some elements.

Definition (Ordering Sets)

i. The set of successors of x $S(x) = \{y \in X \cup Z | y \succeq x\}$.

ii. The set of immediate successors of x $IS(x) = \left\{ y \in S(x) | \ \nexists_{y' \neq y \neq x} y \succeq y' \succeq x \right\}$.

iii. The set of predecessors of x $P(x) = \{y \in X \cup Z | x \succeq y\}$.

iv. The set of immediate predecessors of x $IP(x) = \left\{ y \in P(x) | \ \nexists_{y' \neq y \neq x} x \succeq y' \succeq y \right\}$.

v. The path to a final node z $\text{Path}(z) = \{x \in X | z \succeq x \succeq \alpha\}$.

Definition (Linear Game) An EFG is linear if $\forall_{z\in Z}\forall_v\#\left(\text{Path}\left(z\right)\cap v\right)\leq 1$. where v is an information set of some player.

Definition (Perfect Recall Game) An EFG is of perfect recall if for all i

$$\forall_{v_m^i,v_n^i\in V^i}\forall_{x_1,x_2\in v_n^i,y\in v_m^i}\left[\exists_{c\in C_{v_m^i}}x_1\succeq_c y\to x_2\succeq_c y\right]$$

where $x\succeq_c y$ is read as x comes after y when choosing action c. Another way to write it is that the following does not hold:

$$\exists_{v_m^i,v_n^i\in V^i}\exists_{x_1,x_2\in v_n^i,y\in v_m^i}x_1\succeq_c y\wedge x_2\not\succeq_c y$$

Definition (Perfect Information Game) An EFG is of perfect information if $\forall_i\forall_{v\in V^i}\exists_{x\in P^i}v=\{x\}$. That is if all information sets are singletons.

Definition (Pure Strategy) A pure strategy is a plan of actions for every information set. The set of all pure strategies is $S^i=\{s^i:V^i\to\bigcup C_v|s^i\left(v\right)\in C_v\}$, where $v\in V^i$.

Definition (Payoff from Pure Strategy) Let $u^i:S\to\mathbb{R}$ be the payoff of a pure strategy defined as: $u^i\left(s\right)=\sum\limits_{x\in IS(\alpha)}p\left(x\right)u^i\left(z\left(x,s\right)\right)$ where $p\left(x\right)$ is the probability that node x is played given "nature's" move, and $z\left(x,s\right)$ is the unique final node induced by $x\in IS\left(\alpha\right)$ and the strategy profile s.

Definition (Mixed Strategy) A mixed strategy for player i is $\sigma^i\in\Sigma^i=\Delta\left(S^i\right)$. Note by $\Pr_\sigma$ the probability over final nodes induced by σ.

Definition (Behavioral Strategy) A behavioral strategy $\beta^i\in B$ is a set of functions $\beta^i\left(v,\cdot\right)\in\Delta\left(C_v\right)$, so that $\beta^i=\{\beta^i\left(v,\cdot\right)|\beta^i\left(v,\cdot\right)\in C_v,v\in V^i\}$. Note by $\Pr_\beta$ the probability over final nodes induced by β.

Definition (Induced Normal Form Game) The induced NFG is a tuple $G=\left\{I,\{S^i,u^i\}_{i\in I}\right\}$ where S^i is the set of pure strategies of player i in the EFG and u^i is the payoff from the pure strategies of the EFG.

Definition (NE in Mixed Strategies) A NE in mixed strategies is a NE of the NFG.

Definition (NE in Behavioral Strategies) A NE in behavioral strategies is an strategy β such that for all i:

$$\beta^i\in\text{BR}_B^i\left(\beta\right)=\left\{b^i\in B|\forall_{c\in B}\sum_{z\in Z}u^i\left(z\right)\Pr_{\left(b,\beta^{-i}\right)}\left(z\right)\geq\sum_{z\in Z}u^i\left(z\right)\Pr_{\left(c,\beta^{-i}\right)}\left(z\right)\right\}$$

Definition (Backward Induction) It is assumed that the game if of perfect information and that there will be no ties.

The solution of the backward induction (BI) process is a strategy profile $s = \{s^1, \ldots, s^n\}$ where $s^i\left(x_j^i\right) = c \in C_{x_j^i}$ for $x_j^i \in P^i$.

Let $\Psi = \left\{x_j^i | IS\left(x_j^i\right) \subset Z\right\}$ be the set of "pre-final" nodes, and $\Upsilon = \left\{z \in Z | z \in IS\left(x_j^i\right) \wedge x_j^i \in \Psi\right\}$ the set of final nodes following $x_j^i \in \Psi$. The strategy s is obtained as follows:

i. Arbitrarily take $x_j^i \in \Psi$. Select $c_j^i \in C_{x_j^i}$ such that $u^i\left(z\left(c_j^i\right)\right) \geq u^i\left(z\left(c'\right)\right)$ for all $c' \in C_{x_j^i}$, where $z(c) \in IS\left(x_j^i\right)$ is the final node that follows x_j^i when action c is chosen. set $s^i\left(x_j^i\right) = c_j^i$.

   (a) Note that c exists since $\#C_{x_j^i} < \infty$ for all i, j, and $u^i(z(c)) \neq u^i\left(z\left(c'\right)\right)$ for $c \neq c'$ by assumption.

ii. Repeat the process for all elements of Ψ, this is a finite set since this is a finite game.

iii. Define a new game where $\hat{X} = X \backslash \Psi$, $\hat{Z} = Z \backslash \Upsilon \bigcup \left\{\hat{z}_j^i | \hat{z}_j^i = x_j^i \in \Psi\right\}$ and $\forall_i u^i\left(\hat{z}_j^i\right) = u^i\left(z\left(s^i\left(x_j^i\right)\right)\right)$.

iv. For the new game check $\hat{X}$, if $\hat{X} = \emptyset$ then the game is solved (only nature moves), and the strategy profile s is complete (it has an action for every node of every player). If $\hat{X} \neq \emptyset$ repeat steps (i) to (iii).

Theorem (Zermelo) The Backward Induction procedure is solved in $\#X$ iterations and produces a vector of final payoffs and (at least one) pure strategy profile which is a Nash Equilibrium.

Proposition Every perfect recall game is linear.

Proof: Suppose not and let G be a perfect recall game that is not linear.

- Let $v \in V^i$ be an information set such that there exists a final node z for which $\#\left(\{x \in X | x \in \text{Path}(z) \cap v\}\right) > 1$, this v and z exist since G is not linear.
- Take $x, y \in \text{Path}(z) \cap v$, wlog $x \succeq_c y$ for some action $c \in C_v^i$.
- G is not of perfect recall: let $w = v \in V^i$ and $z = y$ then there exists $c \in C_v^i$ and $x \succeq_c z$ yet $y \succeq_c z$ does not hold since a node cannot come after itself.
- This shows that G is not of perfect recall, which is a contradiction.

Proposition (Behavioral Strategies to Mixed Strategies) Let G be a **linear** EFG and β a behavioral strategy profile. There exists $\sigma \in \Sigma$ a mixed strategy profile such that $\Pr_\sigma = \Pr_\beta$. The mixed strategy is given by $\sigma^i\left(s^i\right) = \prod_{v \in V^i} \beta^i\left(v, s^i(v)\right)$.

Proposition (Mixed Strategies to Behavioral Strategies) Let G be a **perfect recall** EFG and σ a mixed strategy profile. There exists $\beta \in B$ a behavioral strategy profile such that $\Pr_\beta = \Pr_\sigma$. The behavioral strategy is given by $\beta^i(v,c) = \frac{\sum\limits_{s^i \in Rel(v,c)} \sigma^i\left(s^i\right)}{\sum\limits_{s^i \in Rel(v)} \sigma^i\left(s^i\right)}$, where $\mathrm{Rel}(v) = \{s^i \in S^i | \exists_{s^{-i} \in S^{-i}} \mathrm{Path}(s^i, s^{-i}) \cap v \neq \emptyset\}$ is the set of strategies for player i for which there exists a strategy profile that intersects information set v, and $\mathrm{Rel}(v,c) = \{s^i \in \mathrm{Rel}(v) | s^i(v) = c\}$ is the set of those strategies that take action c at v.

Proposition (Best Response in Behavioral Strategies - Theorem 6.2.1 Van Damme pg. 107) A behavioral strategy $b \in B^i$ is a best response to the profile β (note it includes actions of all players) if and only if it is a best response at every $v \in V^i$ that is reached with positive probability under β^{-i}. That is if:

$$\forall_{v \in V^i} \sum_{x \in v} P_\alpha^\beta(x) > 0 \longrightarrow \forall_{b' \in B^i} \sum_{x \in v} \mu(x) \left[\sum_{z \in S(x)} P_x^{\left(b,\beta^{-i}\right)}(z)\, u^i(z)\right] \geq \sum_{x \in v} \mu(x) \left[\sum_{z \in S(x)} P_x^{\left(b',\beta^{-i}\right)}(z)\, u^i(z)\right]$$

where:

i. $P_\alpha^\beta(x)$ is the probability of reaching node x starting from the initial node (α) and given that players move according to β.

ii. $P_x^{\left(b,\beta^{-i}\right)}(z)$ is the probability of reaching final node z given that the initial node is x and players move according to (b, β^{-i}).

iii. $\mu(x) = \frac{P_\alpha^\beta(x)}{\sum_{x \in v} P_\alpha^\beta(x)}$ is the probability of reaching node x given information set v and behavioral strategy β.

Proposition (NE in Behavioral Strategies) A strategy β is a NE in behavioral strategies if and only if β^i is a best response to β at all information sets that are reached by β with positive probability. (That is, if it is a NE in the agent normal form game).

22.1 Subgame Perfect Equilibria

Definition (Subgame) Let G be an EFG and $x \in X_G$ a move node. A subgame G_x is defined from x as follows if x satisfies the consistency condition below.

$G_x = \left\{I, X_{G_x}, Z_{G_x}, \succeq, P_{G_x}, \left\{V^i_{G_x}\right\}_{i \in G_x}, \{u^i\}_{i \in I}\right\}$ where

i. $X_{G_x} = S_G(x) = \{y \in X_G | y \succeq x\}$

ii. $Z_{G_x} = \{z \in Z | z \succeq x\}$

iii. $P^i_{G_x} = P^i_G \cap X_{G_x}$

iv. $V^i_{G_x} = \{v \in V^i_G | v \subseteq S_G(x)\}$

The consistency condition is:

$$\forall_{i \in I} \forall_{v \in V^i} v \subseteq X_{G_x} \vee v \cap X_{G_x} = \emptyset$$

Definition (Behavioral Strategy) Let β^i be a behavioral strategy in G, the restriction β^i_x of β^i is: $\beta^i_x(v, \cdot) = \beta^i(v, \cdot)$ if $v \in V^i_{G_x}$.

Definition (Subgame Perfect Equilibria) A behavioral strategy profile β is a SPE if and only if for all $x \in X$ such that x defines a subgame G_x, the restriction β_x is a NE on G_x.

Proposition (Subgame Perfect Equilibria)

i. The set of SPE is non-empty.

ii. For a game of perfect information a strategy obtained by backward induction is a SPE.

iii. If $\hat{s}$ is a pure strategy NE then $\hat{s}$ induces a NE in all subgames reached by $\hat{s}$. This is different from $\hat{s}$ being a SPE.

22.2 Perfection Under Behavioral and Mixed Strategies

Definition (Perturbation of Behavioral Strategies) $\eta \in \underset{i\in I}{\times}\underset{v\in V^i}{\times}\mathbb{R}^{\#C_v}_{++}$ is a perturbation term, where $\#C_v$ is the number of actions of player i in information set v. $\eta = (\eta^1, \ldots, \eta^n)$ and $\eta^i = \left(\eta^i_1, \ldots, \eta^i_{k^i}\right)$ where k^i is the number of information sets of player i.

Definition (Perturbed Behavioral Strategy) The set of perturbed strategies for player i under perturbation η is $\beta^i_{\eta^i} = \left\{\beta^i \in S^i | \forall_j \forall_{c\in C_{v^i_j}} \beta^i\left(v^i_j, c\right) \geq \eta^i_j > 0\right\}$.

Definition (η-Equilibria) A perturbed strategy $\hat{\beta}(\eta)$ is an η-equilibrium if for all i: $\hat{\beta}(\eta) \in \mathrm{BR}^i_{B^i_{\eta^i}}\left(\hat{\beta}(\eta)\right)$

Definition (Perfect Equilibria in Mixed Strategies) $\sigma \in \Sigma$ is a perfect equilibria for game G if it is a perfect equilibria of the associated NFG.

Definition (Perfect Equilibria in Behavioral Strategies) A behavioral strategy $\beta \in B$ is a perfect equilibrium if there exists a sequence $\{\eta_n\}$ of perturbations such that $\eta_n \to 0$ and a sequence of β_n of behavioral strategies such that for all n β_n is an η_n-equilibrium and $\beta_n \to \beta$.

Note: In general these two concepts do not coincide. They generate different outcomes.

Definition (Agent Normal Form Game) Let G be an extensive form game. Interpret each player as a collection of agents, each for every information set. Each agent has the same payoffs as its player. The agent normal form game is the normal form game associated with this interpretation. The players are given by all the agents, each with the actions of its corresponding information set (C_v).

Proposition (Best Responses and Information Sets) For a player i $b^i \in B$ (a behavioral strategy) is a best response to behavioral profile β if and only if b^i it is a best response to β for each information set $v \in V^i$ that is reached with positive probability when $\beta \backslash b^i$ is played.

Proof: Let $R^i\left(\beta^{-i}\right) = \sum_{z\in Z} P^\beta(z)\, u^i(z)$ is the expected payoff of player i under behavioral profile β. This can be expressed as:

$$\begin{aligned}
R^i(\beta) = \sum_{z\in Z} P^\beta(z)\, u^i(z) &= \sum_{v\in V^i}\left(\sum_{x\in v} P^\beta(x)\right)\sum_{x\in v}\frac{P^\beta(x)}{\sum_{x\in v} P^\beta(x)}\sum_{z\in S(x)}\frac{P^\beta(z)}{P^\beta(x)}u^i(z) \\
&= \sum_{v\in V^i} P^\beta(v)\left[\sum_{x\in v} P^\beta_v(x)\sum_{z\in S(x)} P^\beta_x(z)\, u^i(z)\right] \\
&= \sum_{v\in V^i} P^\beta(v)\, R^i_v(\beta)
\end{aligned}$$

where $P^{\beta}(v)$ is the probability of reaching information set v given β, $P_v^{\beta}(x)$ is the probability of reaching x given information set v and β and $P_x^{\beta}(z)$ is the probability of reaching z given x and β. Finally $R_v^i(\beta)$ is the payoff of the agent associated with information set v given behavioral profile β.

Then b maximizes $R^i(b, \beta\backslash b) = \sum_{v \in V^i} P^{((b,\beta\backslash b))}(v) R_v^i(b, \beta\backslash b)$ if and only if b maximizes $R_v^i(b, \beta\backslash b)$ for each v with $P^{(b,\beta\backslash b)}(v) > 0$.

Corollary β is a η-equilibrium of the EFG if and only if β is a η-equilibrium of the ANF. (in an η-equilibrium all information sets are reached with positive probability).

Proposition (Perfect Equilibria in Behavioral Strategies and Agent Normal Form Game) Behavioral strategy β is a perfect equilibrium of the extensive form game if and only if it is a perfect equilibrium of the agent normal form game.

Note: A perfect equilibria in the agent normal form game always exists.

Proposition (Existence of Perfect Equilibria) Every extensive form game has a perfect equilibria in mixed strategies and a perfect equilibria in behavioral strategies.

22.3 Sequential Equilibria

Definition (Belief) A belief is a function $\mu : X \to [0,1]$ such that $\forall_i \forall_{v \in V^i} \sum\limits_{x \in v} \mu(x) = 1$.

Definition (Rationalization of β) A behavioral strategy β is rational given μ if β^i is a best response to β^{-i} at the subgame defined by every information set. Expected utility is computed at the information set using μ for the nodes that define the information set.

Define the expected utility at information set $v \in V^i$ given that beliefs are μ, player i plays b and other players play β^{-i} is:

$$\sum_{x \in v} \mu(x) \left[\sum_{z \in S(x)} P_x^{\left(b,\beta^{-i}\right)}(z)\, u^i(z) \right]$$

where $P_x^{\left(b,\beta^{-i}\right)}(z)$ is the probability of reaching final node z given that the initial node is x and players move according to behavioral strategy (b, β^{-i}).

Definition (Bayes consistency) A belief μ is Bayes consistent with a behavioral strategy β if:

i. If β is a fully mixed behavioral strategy then μ is defined by Bayes rule using β:

$$\forall_{v \in V^i} \forall_{x \in v} \mu(x) = P_\beta(x|v) = \frac{P_\alpha^\beta(x)}{\sum\limits_{x \in v} P_\alpha^\beta(x)}$$

where $P_\alpha^\beta(x)$ is the probability of reaching node x starting from the initial node and given that player move according to behavioral strategy β.

ii. If β is not a fully mixed behavioral strategy then μ is consistent with β if there exists a sequence $\beta_n \to \beta$ of fully mixed strategies, and a sequence of μ_n consistent with β_n such that $\mu_n \to \mu$.

Definition (Sequential Equilibria) A sequential equilibria is a pair (β, μ) of behavioral strategies and beliefs such that:

i. β is rational at every information set given μ.

ii. μ is Bayes consistent with β

Proposition (Subgame Perfect Equilibria and Sequential Equilibria) Every sequential equilibrium is a subgame perfect equilibrium.

Proof: Let β be a behavioral strategy part of a sequential equilibrium of game G, and suppose that it is not subgame perfect. Then there exists a subgame G_x for which β_x (the restriction of β to G_x) is not a NE. Then it must be that there exists γ_x^i such that for some i:

$$\sum_{z\in Z_x}\left(\prod_{x\in \text{Path}(z)\cap P^i}\gamma_x^i\left(x,c_z\right)\right)\left(\prod_{x\in \text{Path}(z)\setminus P^i}\beta_x\left(x,c_z\right)\right)u^i\left(z\right)>\sum_{z\in Z_x}\left(\prod_{x\in \text{Path}(z)}\beta_x\left(x,c_z\right)\right)u^i\left(z\right)$$

yet this violates β_x being sequential since for all information set of i that is reached with positive probability by β_x the LHS of the inequality above can be expressed in terms of the beliefs induced by β_x. As a response to those beliefs β_x is optimal, then it cannot be that γ_x^i gives a higher payoff.

Proposition (Behavioral Perfect Equilibria and Sequential Equilibria) Every perfect equilibrium is a sequential equilibrium.

Proof: Let $\beta\in B$ be a perfect equilibrium, then there exists a sequence $\{\eta_n\}$ of perturbations such that $\eta_n\to 0$ and a sequence of $\beta_n\in B$ such that for all n β_n is an η_n-equilibrium and $\beta_n\to\beta$.

Since for each n β_n is a fully mixed behavioral strategy define μ_n as the system of beliefs Bayes consistent with β_n.

Note that since β_n is a η_n-equilibrium then it reaches all information sets with positive probability, it must be that β_n^i is a best response to β_n at all information sets:

$$\forall_i\forall_{v\in V^i}\forall_{b\in B^i}\sum_{x\in v}\mu_n\left(x\right)\left[\sum_{z\in S(x)}P_x^{\left(\beta_n,\beta_n^{-i}\right)}\left(z\right)u^i\left(z\right)\right]\geq\sum_{x\in v}\mu_n\left(x\right)\left[\sum_{z\in S(x)}P_x^{\left(b,\beta_n^{-i}\right)}\left(z\right)u^i\left(z\right)\right]$$

that is the definition of β_n being sequentially rational with μ_n.

Define $\mu=\lim\mu_n$, this limit exists since μ_n is a sequence in a compact set. It follows that μ is Bayes consistent with β by construction.

Finally β is sequentially rational with μ. This follows from taking limits on the inequality above (recalling $\beta_n\to\beta$, $\mu_n\to\mu$) and using continuity of $P_x^{(\cdot)}$ (with respect to β_n).

Then (β,μ) is a sequential equilibria.

Proposition (Existence of Sequential Equilibria) Let G be an extensive form game, since a perfect equilibria on behavioral strategies exists for G then a sequential equilibria exists for G.

Note (Types of equilibrium in behavioral strategies)

- Perfect (trembling hand) equilibria makes sure that no weakly dominated strategies are played. Since all information sets (and all nodes in all information sets) are reached with positive probability, weakly dominated strategies are never played.

- Sequential equilibria makes sure that no strongly dominated strategies are played (but there are beliefs under which weakly dominated strategies are played). Beliefs can be such that some nodes in an information set are reached with zero probability, under those beliefs it can be a best response to play weakly dominated strategies.
- Subgame perfect equilibria makes sure that behavior off the equilibrium path is rational.
- Nash equilibria only makes sure that behavior in the equilibrium path is rational.

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