EC9604: Advanced Macroeconomics[1]

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Contents

[1]These notes are intended to summarize the main concepts, definitions and results covered in the core macroeconomics course of the PhD program at the University of Western Ontario, EC9604. These notes only include selected sections of books or articles relevant to the course used here only in part for reference and teaching purposes. Please let me know of any errors that persist in the document.

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Part I

Random Variables, Probability, and Markov Processes

Consider an experiment that can have several (but finite) outcomes. For example trowing a dice can turn out in getting any number from 1 to 6, or asking someone out can generate an affirmative response, a negative one or perhaps a maybe, or no response at all. A probability function is a function that assigns a value to each possible outcome while satisfying certain rules.

Its clear that when the outcomes are finite, outcomes form a set $S = \{s_1, \ldots, s_n\}$, a probability is a list $(\pi_1, \ldots, \pi_n)$ such that $\Pr(s_i) = \pi_i$:

(a) $\pi_i \geq 0$ for all i.

(b) $\sum \pi_i = 1$.

It is natural to define other outcomes that are formed by unions of the former ones, like getting an even number when trowing the dice (the union of getting a two a four and a six) or getting a positive answer or a maybe when asking someone out. It is clear that the probability of these new outcomes is defined by the sum of probabilities of the original outcomes used to define them.

Formally we could say that for any set $A \subseteq S$ we define $I_A = \{i | s_i \in A\}$ and then a function $\mu : 2^S \to [0, 1]$ as:

$$\mu(A) = \Pr(A) = \sum_{i \in I_A} \pi_i$$

Furthermore we can define the expected value of a real valued function $f : S \to \mathbb{R}$ as $E[f] = \sum \mu(\{s_i\}) f(s_i)$.

This same discussion can be carried out if the possible outcomes are countably infinite, but it is difficult to generalize it otherwise. The objective now is to study which properties does this kind of function satisfy and how it is generalized to deal with cases where outcomes are arbitrary. The key for this is to realize that a probability is a function that maps sets into the interval $[0, 1]$, hence the study of functions that map sets into non-negative numbers will provide the necessary theory, these functions are called measures, for obvious reasons.

The following sections draw on the short exposition of measure theory contained in Chapter 7 of Stokey, Lucas, and Prescott (1989) and complements it with portions of

Kolmogorov and Fomin (2012) (chapters 7 to 10). Both these references are introductory although they present all the relevant results. All the material is also covered in a more advanced manner in Kolmogorov and Fomin (1999).

The aim of the course is not to dwell in the mathematical details of the theory but rather present the most useful results for applications in economic theory, because of this many of the proofs will be omitted only including those that are either instructive of the way the theory is developed. Kolmogorov and Fomin (2012) is a good source for detailed (and easy to understand) proofs.

We end with a discussion of Markov processes, defined following Stokey, Lucas, and Prescott (1989, Ch. 8). These processes are key for the stochastic dynamic problems that we will study in the first part of the course

1. Measure

1.1. Measurable spaces (σ-algebras)

Before we define a measure recall that a measure has for domain a collection of sets. For a measure to have some desirable properties this collection of sets cannot be left unrestricted. It turns out that the appropriate family of sets to be consider is that of σ-algebra.

Definition 1.1. (σ-algebra) Let S be a set and $\mathcal{A} \subseteq 2^S$ a family of its subsets. $\mathcal{A}$ is a σ-algebra if and only if:

(a) $\emptyset, S \in \mathcal{A}$.

(b) $A \in \mathcal{A}$ implies $A^c = S \backslash A \in \mathcal{A}$. We say that $\mathcal{A}$ is closed under complement.

(c) $A_n \in \mathcal{A}$ for $n = 1, \ldots$ implies $\cup A_n \in \mathcal{A}$. We say that $\mathcal{A}$ is closed under countable union.

(i) $\mathcal{A}$ is closed under countable intersection because $\cap A_n = \left(\cup A_n^c\right)^c$.

If $\mathcal{A}$ is only closed under finite union (or intersection) then $\mathcal{A}$ is an algebra.

A σ-algebra imposes certain consistency to the family of sets under consideration. The way to interpret it is that only subsets of the σ-algebra can be known, hence measured. Because of property (i) it is possible to know when none or all of the outcomes occurred. Also if there is an outcome that occurred it must be possible to determine if it didn't. Finally if it is possible to determine that some outcomes occurred individually it can also be determined if at least one or all of them were realized.

It is instructive to consider two simple examples of σ-algebras that arise from throwing a 4 sided dice, then $S = \{1, 2, 3, 4\}$. One (trivial) σ-algebra is:

$$\mathcal{A} = \left\{\emptyset, S\right\}$$

Another one is the σ-algebra generated by the collection $\{\{1\}, \{2\}, \{3\}, \{4\}\}$, then:

$$\mathcal{A} = \left\{ \begin{array}{c} \{1\}, \{2\}, \{3\}, \{4\}, \{2, 3, 4\}, \{1, 3, 4\}, \{1, 2, 4\}, \{1, 2, 3\}, \\ \{1, 2\}, \{2, 3\}, \{3, 4\}, \{1, 3\}, \{1, 4\}, \{2, 4\}, \emptyset, S \end{array} \right\}$$

In this case $\mathcal{A} = 2^S$, but this is not necessarily true, imagine that one can only determine if an even number was thrown, then the outcomes are $\{\{1, 3\}, \{2, 4\}\}$, the σ-algebra is:

$$\mathcal{A} = \left\{\{1, 3\}, \{2, 4\}, \emptyset, S\right\}$$

When S has uncountably many elements this process cannot be exemplified as easily but one can always define the σ-algebra generated by a subset $\mathcal{A} \subseteq 2^S$ as the intersection of all σ-algebras that contain $\mathcal{A}$. Clearly the arbitrary intersection of σ-algebras is again a σ-algebra.

Now that we have defined a σ-algebra its possible to say what a measurable set and a measurable space are:

Definition 1.2. (Measurable Space) A pair $(S, \mathcal{A})$ where S is any set and $\mathcal{A}$ is a σ-algebra is called a measurable space. A set $A \in \mathcal{A}$ is called $\mathcal{A}$-measurable.

We say that $A \subseteq S$ is measurable with respect to a σ-algebra $\mathcal{A}$ if its elements are identifiable, that is, if the outcomes represented in A can be told apart from other outcomes given the information in $\mathcal{A}$. For example the set $A = \{4\}$ is not measurable in the last example above, because its impossible to know if a 4 was the outcome of the throw.

A σ-algebra of special importance is the Borel σ-algebra.

Definition 1.3. (Borel σ-algebra) Let $S = \mathbb{R}$ and $\mathcal{A}$ be the set of open and half open intervals. The Borel algebra, noted by $\mathcal{B}$, is the σ-algebra generated by $\mathcal{A}$. A set $B \in \mathcal{B}$ is called a Borel set.

The Borel algebra could have been defined equivalently with the closed and half closed intervals (use complement). In general one can define the Borel algebra for any metric space (S, ρ) as the smallest σ-algebra containing all the open balls. In the case of the Euclidean spaces it can also be generated with open rectangles.

What follows is to define the measure of a measurable set.

1.2. Measures

1.2.1. Measures in σ-algebras

Given a measurable space $(S, \mathcal{A})$ a measure is nothing but a function $\mu : \mathcal{A} \to \overline{\mathbb{R}}$ with certain restrictions that guarantee its consistency:

Definition 1.4. (Measure) Let $(S, \mathcal{A})$ be a measurable space. A measure is an extended real-valued function $\mu : \mathcal{A} \to \overline{\mathbb{R}}$ such that:

(a) $\mu(\emptyset) = 0$

(b) $\mu(A) \geq 0$ for all $A \in \mathcal{A}$.

(c) μ is countably additive. If $\{A_n\}_{n=1}^{\infty}$ is a countable, disjoint sequence in $\mathcal{A}$, then:

$$\mu(\cup A_n) = \sum \mu(A_n)$$

If furthermore $\mu(S) < \infty$ then μ is said to be a finite measure, and if $\mu(S) = 1$ then μ is said to be a probability measure.

Definition 1.5. (Measure Space) A triple $(S, \mathcal{A}, \mu)$ where S is a set, $\mathcal{A}$ is a σ-algebra of its subsets and μ is a measure on $\mathcal{A}$ is called a measure space. The triple is called a probability space if μ is a probability measure.

An important concept is that of almost everywhere and almost surely. These are qualifiers for a given proposition that can be evaluated in sets of $\mathcal{A}$.

Definition 1.6. (Almost Everywhere and Almost Surely) Let $(S, \mathcal{A}, \mu)$ be a measure space. A proposition is said to hold almost everywhere (a.e.) or almost surely (a.s.) if there exists a set $A \in \mathcal{A}$ such that $\mu(A) = 0$ and the proposition holds in A^c.

An example of the use of a.e. or a.s. is when treating functions that are similar to each other. One can say that two functions are equivalent a.e. or that a function is continuous a.e. Then the functions f and g satisfy $f(x) = g(x)$ and $A = \{x | f(x) \neq f(y)\}$ satisfies $\mu(A) = 0$. In measure theory the behavior of functions a.e. is all that matters, then we can treat functions that have anomalies as long as those anomalies occur only in sets of measure zero.

There are some properties of a measure that are useful to keep in mind, a crucial one is used for Bayes law and the definition of conditional probability.

Proposition 1.1. *Let* $(S, \mathcal{A}, \mu)$ *be a measure space and* $B \in \mathcal{A}$ *a set. Define* $\lambda : \mathcal{A} \to \overline{\mathbb{R}}$ *as* $\lambda(A) = \mu(A \cap B)$*. Then* λ *is a measure on* $(S, \mathcal{A})$*. If in addition* $\mu(B) < \infty$ *then* $\tilde{\lambda}$ *defined as* $\tilde{\lambda}(A) = \mu(A \cap B)/\mu(B)$ *is a probability measure on* $(S, \mathcal{A})$*.*

Proof. First, if $A, B \in \mathcal{A}$ then $A \cap B \in \mathcal{A}$, this follows from a σ-algebra being closed under countable intersection, by letting $A_1 = A$ and $A_n = B$ for $n \geq 2$ the result obtains. It is left to check the three properties of a measure:

(a) $\lambda(\emptyset) = \mu(\emptyset \cap B) = \mu(\emptyset) = 0$.

(b) $\lambda(A) = \mu(A \cap B) \geq 0$.

(c) Let $\{A_n\}_{n=1}^{\infty}$ be a countable, disjoint sequence in $\mathcal{A}$, then the sequence $\{A_n \cap B\}_{n=1}^{\infty}$ is also disjoint and that:
$$\lambda(\cup A_n) = \mu((\cup A_n) \cap B) = \mu(\cup (A_n \cap B)) = \sum \mu(A_n \cap B) = \sum \lambda(A_n)$$

(d) If $\mu(B) < \infty$ then all the previous results hold for $\tilde{\lambda}$ by dividing everything by $\mu(B)$. Furthermore $\tilde{\lambda}(S) = \frac{\mu(S \cap B)}{\mu(B)} = \frac{\mu(B)}{\mu(B)} = 1$.

□

Another useful property is given by the following proposition, it reflects the intuitive property of measures being 'increasing':

Proposition 1.2. *Let* $(S, \mathcal{A}, \mu)$ *be a measure space and* $A, B \in \mathcal{A}$ *sets. If* $A \subseteq B$ *then* $\mu(A) \leq \mu(B)$*, if in addition* μ *is finite then* $\mu(B \backslash A) = \mu(B) - \mu(A)$*.*

Proof. Because $A \subseteq B$, there exits $C = B \backslash A = B \cap A^c$ such that $A \cup C = B$ and $A \cap C = \emptyset$. Then
$$\mu(A) + \mu(C) = \mu(B).$$
Moreover, $\mu(A) \leq \mu(B)$ because $\mu(C) \geq 0$. If μ is finite then all elements above are well defined and: $\mu(B \backslash A) = \mu(B) - \mu(A)$.

□

The following property is widely used to establish properties of limits of functions, and of the Lebesgue integral:

Proposition 1.3. *Let* $(S, \mathcal{A}, \mu)$ *be a measure space:*

(a) *If* $\{A_n\}$ *is an increasing sequence in* $\mathcal{A}$*, that is, if* $A_n \subseteq A_{n+1}$ *for all n, then:*
$$\mu(\cup A_n) = \lim \mu(A_n)$$

(b) *If $\{B_n\}$ is an decreasing sequence in $\mathcal{A}$, that is, if $B_n \supseteq B_{n+1}$ for all n, then:*

$$\mu(\cap B_n) = \lim \mu(B_n)$$

Proof. Stokey, Lucas, and Prescott (1989, Sec. 7.2). Satisfying these two properties makes a measure continuous.

□

1.2.2. Measures in algebras and extensions [Optional]

So far we have defined a measure on an σ-algebra, but a σ-algebra is usually a large collection of sets and defining a function on such a set while preserving the consistency required for a measure is not an easy task. An alternative is given by defining measures on algebras, which are smaller and less complicated collections of sets. It can be shown that these measures preserve all the desirable properties of the more complicated spaces, and also allow for an extension to σ-algebras, once the measure is properly constructed.

We start by defining a measure on an algebra.

Definition 1.7. (Measure) Let $(S, \mathcal{A})$ be a measurable space. A measure is an extended real-valued function $\mu : \mathcal{A} \to \overline{\mathbb{R}}$ such that:

(a) $\mu(\emptyset) = 0$

(b) $\mu(A) \geq 0$ for all $A \in \mathcal{A}$.

(c) If $\{A_n\}_{n=1}^{\infty}$ is a countable, disjoint sequence in $\mathcal{A}$, and $\cup A_n \in \mathcal{A}$, then:

$$\mu(\cup A_n) = \sum \mu(A_n)$$

If furthermore $\mu(S) < \infty$ then μ is said to be a finite measure, and if $\mu(S) = 1$ then μ is said to be a probability measure.

Condition (iii) also includes finite union of disjoint sets as a special case.

Definition 1.8. (σ-finite measure) Let S be a set, $\mathcal{A}$ an algebra of its subsets and μ a measure defined on $\mathcal{A}$. If there is a countable sequence of sets in $\mathcal{A}$, $\{A_n\}$, such that $\mu(A_n) \leq \infty$ and $S = \cup A_n$ then μ is σ-finite

It is now possible to extend the notion of this measure to a σ-algebra.

Theorem 1.1. (Caratheodory extension theorem) Let S be a set, $\mathcal{A}$ an algebra of its subsets and μ a measure defined on $\mathcal{A}$. Let $\mathcal{A}^\star$ be the smallest σ-algebra containing $\mathcal{A}$. There exists a measure $\mu^\star$ on $\mathcal{A}^\star$ such that $\mu^\star(A) = \mu(A)$ for all $A \in \mathcal{A}$.

The problem of uniqueness is also solved.

Theorem 1.2. (Hahn extension theorem) Let S be a set, $\mathcal{A}$ an algebra of its subsets, μ a measure defined on $\mathcal{A}$ and $\mathcal{A}^\star$ the minimal σ-algebra of $\mathcal{A}$. If μ is σ-finite then the extension $\mu^\star$ is unique.

To see how these theorems and the extension of a measure are used consider defining a measure on the Borel σ-algebra. It seems logical to define the measure of an interval $A = (a, b)$ as $\mu(A) = b - a$ if $b \geq a$ and $\mu(A) = 0$ otherwise (because the interval would be empty). Yet the Borel σ-algebra contains sets beyond simple intervals, and the countable union of intervals can give rise to weird sets. An answer to this problem is given by defining a measure on the Borel algebra, formed by all types of intervals and their finite unions. Defining a measure on this set seems straightforward:

(a) $\mu(\emptyset) = 0$

(b) $\mu((a, b)) = \mu\left(\left[a, b\right]\right) = \mu\left((a, b]\right) = \mu\left(\left[a, b)\right) = b - a$

(c) $\mu((-\infty, \infty)) = \mu\left((-\infty, b]\right) = \mu([a, \infty)) = \infty$

(d) $\mu(\cup(a_n, b_n)) = \sum(b_n - a_n)$ if the intervals are disjoint.

The function μ can be verified to be a measure on the Borel algebra, and hence an extension to the Borel σ-algebra exists. If we restrict our attention to $S = \left[a, b\right]$ and the intervals contained in it we can define a σ-finite measure, obtaining uniqueness of the extension. This is how we can deal with complicated environments.

Once the measure is extended to the σ-algebra all the results obtained above apply.

1.2.3. Completion of a measure [Optional]

One small detail is left to be checked. Sometimes there is a set $B \subseteq S$ such that $B \subseteq A \in \mathcal{A}$ and $\mu(A) = 0$, but if $B \notin \mathcal{A}$ then its measure is undefined, while it should be clearly zero. The completion of a σ-algebra to include these type of 'harmless' sets is what follows. As before, including sets or behaviors of measure zero is of no consequence.

Definition 1.9. (Completion of a σ-algebra) Let $(S, \mathcal{A}, \mu)$ be a measure space. Define a collection $\mathcal{C}$ as:

$$\mathcal{C} = \left\{C \subset S | \exists_{A \in \mathcal{A}} \mu(A) = 0 \quad \wedge \quad C \subset A\right\}$$

The completion of σ-algebra $\mathcal{A}$ is:

$$\mathcal{A}' = \left\{ B' \subseteq S \mid B' = (A \cup C_1) \setminus C_2 \qquad A \in \mathcal{A} \quad \wedge \quad C_1, C_2 \in \mathcal{C} \right\}$$

By letting $C_1 = C_2 = \emptyset$ we get $\mathcal{A} \subseteq \mathcal{A}'$, $\mathcal{A}'$ includes all sets in 2^S that differ from a set in $\mathcal{A}$ by a set of measure 0.

Definition 1.10. (Completion of a measure) Let $(S, \mathcal{A}, \mu)$ be a measure space and $\mathcal{A}'$ the completion of $\mathcal{A}$. $\mu\left(B'\right) = \mu(B)$ for any $B' \in \mathcal{A}'$ that differs from $B \in \mathcal{A}$ by a set of measure 0.

The Caratheodory and Hahn extension theorems also apply for completions.

2. Measurable functions

A measurable function is a type of function for which it is possible to know (to measure) the conditions (the set) that originates certain outcomes. One can think of a function as mapping certain events in a given measure space to outcomes in another measure space. A function is measurable if the sets that induce a given outcome are measurable. Formally:

Definition 2.1. (Measurable function) Let $(S, \mathcal{A}, \mu)$ and $\left(S', \mathcal{A}', \mu'\right)$ be measure spaces and $f : S \to S'$ a function. f is measurable if and only if $f^{-1}\left(A'\right) \in \mathcal{A}$ for all $A' \in \mathcal{A}'$.

A special case of notable importance is that of $\left(S', \mathcal{A}', \mu'\right) = (\mathbb{R}, \mathcal{B}, \lambda)$, where λ is the Lebesgue measure on the plane. This are real valued functions. In this case the $\mathcal{B}$-measurable sets in $\mathbb{R}$ can be characterized in the following way:

Theorem 2.1. *Let $(S, \mathcal{A}, \mu)$ be a measure space and $f : S \to \mathbb{R}$. f is μ-measurable if and only if $f^{-1}((-\infty, c)) = \{x \in S | f(x) < c\} \in \mathcal{A}$ for all $c \in \mathbb{R}$.*

Proof. This theorem is stated as the definition of a real valued function f being μ-measurable in Stokey, Lucas, and Prescott (1989), but a formal proof is presented in Kolmogorov and Fomin (2012, Sec. 28, Thm. 1). It can also be stated with any of the inequalities $\geq, \leq, >, <$.

□

Also when the measure space in question is a probability space one can characterize formally what a random variable is.

Definition 2.2. (Random variable) Let $(S, \mathcal{A}, P)$ be a probability space and $f : S \to \mathbb{R}$ a real valued function. f is a random variable if and only if f is measurable, that is, if and only if $f^{-1}(B) \in \mathcal{A}$ for all $B \in \mathcal{B}$, where $\mathcal{B}$ is the Borel σ-algebra on $\mathbb{R}$. We further establish the same notation:

(a) An outcome is an element $s \in S$.

(b) An event is a measurable subset of S: $A \in \mathcal{A}$.

(c) The real number $f(s)$ is a realization of the random variable.

(d) The probability measure for f is then: $\mu(B) = P\left(f^{-1}(B)\right) = P\left(\{s \in S | f(s) \in B\}\right)$, for $B \in \mathcal{B}$.

(e) The distribution function for f is: $G(b) = \mu\left((-\infty, b]\right)$, for $b \in \mathbb{R}$.

Generally it is very hard to find a function that is not measurable. The details of the example will depend on the spaces considered. For example if $f : \mathbb{R} \to \mathbb{R}$ and $\mathcal{A}$ is the set of all open (or closed) sets in $\mathbb{R}$ the definition of measurability is equivalent to that of continuity (the pre-image of an open set has to be open) and then all functions that are not continuous are not measurable. It is clear that more complete σ-algebras make more difficult to generate counterexamples. The following three results show how difficult it is to generate them:

Proposition 2.1. *Let $f : \mathbb{R} \to \mathbb{R}$.*

(a) *If f is continuous then f is measurable with respect to the Borel sets.*

(b) *If f is monotone then f is measurable with respect to the Borel sets.*

Proof. Each case is proven:

(a) Let f be continuous. Consider the set $f^{-1}((-\infty, c))$ for any $c \in \mathbb{R}$. The set $(-\infty, c)$ is open, because f is continuous then its pre-image is open, then it is a Borel set. Then its measurable.

(b) Let f be monotone increasing. Consider the set $f^{-1}((-\infty, c))$ for arbitrary $c \in \mathbb{R}$. Then, $f^{-1}((-\infty, c)) = (-\infty, a)$ or $f^{-1}((-\infty, c)) = (-\infty, a]$ or $f^{-1}((-\infty, c)) = (-\infty, \infty)$ or $f^{-1}((-\infty, c)) = \emptyset$ for some $a \in \mathbb{R}$. Monotonicity ensures that if $a \in f^{-1}((-\infty, c))$ and $b \leq a$ then $b \in f^{-1}((-\infty, c))$. Suppose its not, then there exists numbers $b \leq a$ such that $f(b) > c \geq f(a)$, contradicting monotonicity.

All these sets are in $\mathcal{B}$, then f is $\mathcal{B}$-measurable.

□

Corollary 2.1. *The composition of measurable functions is measurable. In particular the composition of a continuous function with a measurable function is measurable.*

Proposition 2.2. *Let $S = \{s_1, s_2, \ldots\}$ be a countable set (potentially infinite) and $\mathcal{A} = 2^S$ a σ-algebra on S. Then all functions $f : S \to \mathbb{R}$ are measurable.*

Proof. The proof is immediate because the pre-image of a Borel set is a subset of S, then it belongs to $\mathcal{A} = 2^S$.

□

In a more general way one can establish the measurability of a function by relating to a class of well behave 'simple' functions. The base for this class is the indicator function.

Definition 2.3. (Indicator Function) Let $(S, \mathcal{A})$ be a measurable space. An indicator function $\chi_A : S \to \mathbb{R}$ is:

$$\chi_A(s) = \begin{cases} 1 & \text{if } s \in A \\ 0 & \text{if } s \notin A \end{cases}$$

Clearly χ_A is measurable if and only if $A \in \mathcal{A}$.

Definition 2.4. (Simple Function) Let $(S, \mathcal{A})$ be a measurable space. A simple function is a function that takes at most countably many values. When the function takes finitely many values it can be expressed as:

$$\phi(s) = \sum_{i=1}^{n} \alpha_i \chi_{A_i}(s)$$

where $\{A_i\}$ is a sequence of subsets of S and $\alpha_i \in \mathbb{R}$.

Characterizing the measurability of simple functions is slightly more complicated.

Proposition 2.3. *A simple function taking values $\{y_1, y_2, \ldots\}$ is measurable if and only if the sets $A_i = \{s \in S | \phi(s) = y_n\}$ are measurable.*

Proof. Both directions are proven.

(a) Let ϕ be measurable, and $\{y_n\} \in \mathcal{B}$, then its pre-image is measurable wrt $\mathcal{A}$.

(b) Let the sets be measurable, that is$A_i \in \mathcal{A}$, and consider $B \in \mathcal{B}$ a Borel set. Then

$$\phi^{-1}(B) = \{s \in S | \phi(s) = y_i \in B\} = \bigcup_{y_i \in B} A_i.$$

Because each $A_i \in \mathcal{A}_i$ and the union is taken over no more than countably many sets we have $\bigcup_{y_i \in B} A_i \in \mathcal{A}$ by definition of a σ-algebra. This proves measurability of $\phi^{-1}(B)$.

□

In what follows all simple functions will be considered measurable. The importance of simple functions is given by the applications of the following proposition.

Proposition 2.4. *Let $(S, \mathcal{A})$ be a measurable space and let $\{f_n\}$ be a sequence of measurable functions converging pointwise to f, that is* $\lim f_n(s) = f(s)$ *for all s. Then f is also measurable.*

Proof. The proof can be found in Stokey, Lucas, and Prescott (1989, Sec. 7.3) or in Kolmogorov and Fomin (2012, Sec. 28.1).

□

Corollary 2.2. *If f is non-negative one can choose the sequence $\{f_n\}$ to be strictly increasing.*

Corollary 2.3. *If f is bounded one can choose the sequence $\{f_n\}$ to converge uniformly.*

The main application is the following result that gives a characterization of measurable functions in terms of simple functions:

Proposition 2.5. *A function f is measurable if and only if it an be represented as the limit of a uniformly converging sequence of measurable simple functions.*

Proof. The first direction is immediate from the previous proposition. If f is the limit of measurable functions then f is also measurable.

Let f be measurable. It is left to construct a converging sequence of simple functions that converges to f. wlog let $f(s) \geq 0$ for all s, then by the Archimedean principle there exists a non-negative integer m such that

$$\frac{m}{n} \leq f(s) < \frac{m+1}{n}$$

Let $f_n(s) = m/n$, because n is fixed and $m \in \mathbb{N} \cup \{0\}$ it follows that f_n can take at most countably many values, hence it is simple. f_n is also measurable because

$$f_n^{-1}((-\infty, c)) = \{s \in S | f_n(s) \leq c\} = \left\{s \in S | f_n(s) \leq \frac{m^\star}{n}\right\} = \left\{s \in S | f_n(s) < \frac{m^\star + 1}{n}\right\}$$

For $m^\star$ chosen by the Archimedean principle. The last set is $f^{-1}\left(\left(-\infty, \frac{m^\star+1}{n}\right)\right)$ which is measurable by assumption. Then f_n is measurable for all n.

Finally, $f_n \to f$ uniformly because

$$|f_n(s) - f(s)| \leq \left|\frac{m}{n} - \frac{m+1}{n}\right| = \frac{1}{n}.$$

□

Other results will follow and are left stated without proof:

Proposition 2.6. *Let f, g be measurable functions and $\alpha \in \mathbb{R}$ then:*

(a) *$f + g$ is measurable.*

(b) *αf is measurable.*

(c) *fg is measurable.*

(d) *$1/f$ is measurable provided that $f(s) \neq 0$.*

Finally continuity of functions is used to strengthen the intuition around measurability.

Proposition 2.7. *Let f, g be equivalent function defined on an interval E, that is they are equal a.e. If f and g are continuous then they coincide.*

Proof. Suppose not, then there exists $x \in E$ such that $f(x) \neq g(x)$. Let $\epsilon = |f(x) - g(x)|$, because f and g are continuous there exists δ such that for $x' \in B_\delta(x)$ it holds that $\left|f(x) - f\left(x'\right)\right| < \frac{\epsilon}{2}$ and $\left|g(x) - g\left(x'\right)\right| < \frac{\epsilon}{2}$. Then for all $x' \in B_\delta(x)$ it holds that $f\left(x'\right) \neq g\left(x'\right)$, but $B_\delta(x)$ has strictly positive measure, contradicting f and g being equivalent.

□

Proposition 2.8. *A function f equivalent to a measurable function g is measurable.*

Proof. Because the functions are equivalent the sets $\{x|f(x) \leq c\}$ and $\{x|g(x) \leq c\}$ can differ in at most by a set of measure zero. Then if the second set is measurable so is the first one (taking into account the completion of the σ-algebra). This proves measurability.

□

Corollary 2.4. *A function f equivalent to a continuous function is measurable.*

Proof. Immediate from continuous functions being measurable.

□

This implies that if a function is continuous a.e. then it is measurable, again the behavior of functions in sets of measure zero carries no consequence. It turns out that this corollary can be strengthened. The result is powerful and is stated without a proof:

Theorem 2.2. ***(Luzin)*** *Let $f : [a, b] \to \mathbb{R}$ be a function. f is measurable if and only if for all $\epsilon > 0$ there exists a continuous function g such that $\mu\{x \in [a, b] \mid f(x) \neq g(x)\} < \epsilon$.*

This theorem shows that for the case of functions of real variable and real value measurability is equivalent to continuity, except on a set of arbitrarily small size. In other words a measurable function can be made continuous by altering its values on a set of arbitrarily small measure.

3. The Lebesgue integral [Optional]

The Lebesgue integral is in at least two important ways a generalization of the Riemann integral and it serves a crucial purpose of defining what it means to take the expected value of a function with respect to a probability distribution. The first sense in which the Riemann integral is generalized is that the Lebesgue integral is defined over measurable functions, a space that is much richer than that of Riemann integrable functions, the second sense is much more crucial: the Lebesgue integral is defined for functions with domain in arbitrary sets, thus allowing to handle a more abstract and general class of functions.

Intuitively the Lebesgue integral is constructed in a similar way than the Riemann integral. To construct the latter one takes successively finer grids of the domain and evaluate the function at certain points, constructing step functions, one above the function and one below, then two sums are constructed and the value of the integral is defined as the (common) value of the limit of those sums as the length of the grid's spaces goes to zero.

The Lebesgue integral of a function $f : S \to \mathbb{R}_+$ is constructed by taking grids over the range of the function $\{y_i\}_{i=1}^n$ such that $0 = y_1 \leq \ldots \leq y_n$. Then one can define the sets $A_i = \{s \in S | y_i \leq f(s) < y_{i+1}\}$ and using the measure over S define $\lambda(A_i)$ and the sum $\sum y_i \lambda(A_i)$. The Lebesgue integral is then the limit of this sum as the values y_i are closer together.

The introduction before of simple functions makes sense when defining the Lebesgue integral. Its definition seems intuitive for this class of functions and Proposition 2.5 creates a bridge between them and the more general class of measurable functions, thus allowing to extend the Lebesgue integral to this broader family.

In what follows we restrict attention to non-negative, real valued functions.

Definition 3.1. (Lebesgue integral for simple functions) Let $(S, \mathcal{A}, \mu)$ be a measure space and $f : S \to \mathbb{R}_+$ a simple, μ-measurable function that takes no more than countably many values $\{y_1, y_2, \ldots\}$. The Lebesgue integral over the set $A \subseteq S$ is defined as:

$$\int_A f(s)\, d\mu = \sum_n y_n \mu(A_n) \tag{3.1}$$

where the sets A_n are defined as:

$$A_n = \{s \in A | f(s) = y_n\}$$

These sets can be empty if there is no element of s in A for which f takes a given value. The Lebesgue integral is defined as long as the series in (3.1) is absolutely convergent. If f takes finitely many values and μ is finite (or a probability measure) this condition is satisfied.

An example is given by the constant function, $f(s) = 1$ for all $s \in S$, then:

$$\int_A f(s)\, d\mu = \int_A d\mu = \mu(A)$$

It can be shown that the lebesgue integral satisfies some natural properties:

Proposition 3.1. *Let f and g be non-negative, measurable, simple and integrable functions on $(S, \mathcal{A}, \mu)$, a measure space, and $c \geq 0$ a constant. Then:*

(a) $\int_A (f+g)(s)\, d\mu = \int_A f(s)\, d\mu + \int_A g(s)\, d\mu$

(b) $\int_A (cf)(s)\, d\mu = c \int_A f(s)\, d\mu$

(c) *If f is bounded $|f(s)| \leq M$ a.e. then f is integrable and $\left|\int_A f(s)\, d\mu\right| \leq M\mu(A)$.*

Proof. Kolmogorov and Fomin (2012, Sec. 29.1).

□

Definition 3.2. (Lebesgue integral - Nonnegative functions) Let $(S, \mathcal{A}, \mu)$ be a measure space. A measurable function $f : S \to \mathbb{R}$ is said to be integrable on a set A if there exists a sequence $\{f_n\}$ of integrable simple functions converging uniformly to f on A. The Lebesgue integral is defined as:

$$\int_A f(s)\, d\mu = \lim \int_A f_n(s)\, d\mu \tag{3.2}$$

This definition precludes the integral from being infinite, as shown in Kolmogorov and Fomin (2012, Sec. 29.1), the limit above exists provided that the functions f_n are integrable (recall that it was asked of the sum in (3.1) to be finite), moreover it is independent of the choice of sequence approximating f, this sequence can be furthermore be chosen to be strictly increasing (Stokey, Lucas, and Prescott 1989). Yet, the concept of the Lebesgue integral can be easily generalized to allow for infinite values, the definition in Stokey, Lucas, and Prescott (1989) allows for this.

What follows is a list of properties of the Lebesgue integral which should be familiar if there is any knowledge of the behavior of Riemann integrals. They are not of particular interest in this course.

Proposition 3.2. *Properties of the Lebesgue integral for non-negative measurable functions:*

(a) $\int_A (f+g)(s)\,d\mu = \int_A f(s)\,d\mu + \int_A g(s)\,d\mu$

(b) $\int_A (cf)(s)\,d\mu = c\int_A f(s)\,d\mu$

(c) *If g is measurable and integrable and f is bounded by g: $|f(s)| \leq g(s)$ a.e., then f is integrable and $\left|\int_A f(s)\,d\mu\right| \leq \int_A g(s)\,d\mu$.*

  (i) *If f is bounded and measurable then it is integrable.*

(d) *If $f \leq g$ a.e. then $\int f(s)\,d\mu \leq \int g(s)\,d\mu$.*

(e) *If $A \subseteq B$ with $A, B \in \mathcal{A}$ then $\int_A f(s)\,d\mu \leq \int_B f(s)\,d\mu$*

(f) *Let $A = \cup A_n$ where $\{A_n\}$ is a finite or countable sequence of disjoint sets. If f is integrable on A then f is integrable on A_n for all n and:*

$$\int_A f(s)\,d\mu = \sum_n \int_{A_n} f(s)\,d\mu$$

when the series on the right is absolutely convergent.

Finally it is noted that a non-negative integrable function induces a measure on a space, the following proposition makes this clear.

Proposition 3.3. *Let f be a non-negative, integrable function, then $\lambda : \mathcal{A} \to \mathbb{R}$ defined as:*

$$\lambda(A) = \int_A f(s)\,d\mu$$

is a measure on $(S, \mathcal{A})$.

Definition 3.3. (Lebesgue integral) Let $(S, \mathcal{A}, \mu)$ be a measure space. A measurable function $f : S \to \mathbb{R}$ is said to be integrable if the following two integrals are finite:

$$\int f^+(s)\,d\mu \qquad \int f^-(s)\,d\mu$$

where:

$$f^+(s) = \begin{cases} f(s) & \text{if } f(s) \geq 0 \\ 0 & \text{if } f(s) < 0 \end{cases} \qquad f^+(s) = \begin{cases} 0 & \text{if } f(s) \geq 0 \\ -f(s) & \text{if } f(s) < 0 \end{cases}$$

The integral of f is defined as:

$$\int f(s)\, d\mu = \int f^{+}(s)\, d\mu - \int f^{-}(s)\, d\mu \tag{3.3}$$

Recall that when $(S, \mathcal{A}, \mu)$ is a probability space the function f is called a random variable, the definitions above are then the definitions of the expected value of a random variable, this expected value exists when f is integrable, we have seen that a sufficient condition for this is to be bounded a.e. and the measure to be finite, this last condition is satisfied immediately by probability measures.

4. The Stieltjes integral [Optional]

The Lebesgue-Stieltjes integral is a type of integral specially useful in probability theory, because of the resemblance between the Stieltjes measures and probability measures. To introduce the concept consider a real valued random variable that takes values on a closed interval $[a, b]$, this is for example the result of coin toss when catalogued as 0 or 1, the underlying probability space is formed by $S = \{H, T\}$, $\mathcal{A} = \left\{\emptyset, S, \{H\}, \{T\}\right\}$ and a probability measure on $\mathcal{A}$, a function $\mu : \mathcal{A} \to [0, 1]$ such that $\mu(\{H\}), \mu(\{T\}) \geq 0$, $\mu(S) = \mu(\{H\}) + \mu(\{T\}) = 1$ and $\mu(\emptyset) = 0$. The random variable is then a function $f : S \to \mathbb{R}$ such that $f(H) = 0$ and $f(T) = 1$. It seems natural to ask what is the probability that $f(s) = 1$, it is of course given by $\mu(T)$, in the same way can ask for the probability that $f(s) \leq c$ for any value c, the function that answers that question is called the cumulative distribution function. In this example we have:

$$F(c) = \Pr(f(s) \leq c) = \begin{cases} 0 & \text{if } c < 0 \\ \mu(H) & \text{if } 0 \leq c < 1 \\ 1 & \text{if } 1 \leq c \end{cases}$$

Because the measure μ is non-negative it is clear that F has to be a non-decreasing function, it is also continuous from the left, moreover it is possible to recover μ from knowledge of F:

$$\mu(H) = F(0) \qquad \mu(T) = 1 - F(0)$$

The Stieltjes measure is a general way of looking at this last step. It treats the problem of inducing a measure from a non-decreasing left continuous function. The application to probability theory is apparent because we deal with the CDF of a random variable, and not directly with its probability measure, as we saw before it is this latter object the one that defines the expected value.

Now we turn to define formally the Stieltjes integral. Let $F : [a, b] \to \mathbb{R}$ be a non-decreasing and left-continuous function. Let $\mathcal{A}$ be an algebra of all subintervals of $[\alpha, \beta)$ (including open, closed and half-open intervals). Define a measure on $\mathcal{A}$ by:

$$\begin{aligned} m(\alpha, \beta) &= F(\beta) - F(\alpha + 0) \\ m[\alpha, \beta] &= F(\beta + 0) - F(\alpha) \\ m(\alpha, \beta] &= F(\beta + 0) - F(\alpha + 0) \\ m[\alpha, \beta) &= F(\beta) - F(\alpha) \end{aligned}$$

Now consider the Lebesgue extension of m, call it μ_F and the σ-algebra of all μ_F-measurable, call it $\mathcal{A}_F$. This set contains all subintervals of $[\alpha, \beta)$ and hence all the Borel sets of $[\alpha, \beta)$.

Definition 4.1. (Stieltjes measure) The measure μ_F described above is called the (Lebesgue-)Stieltjes measure and F its generating function.

This concept is easily extended to the whole real line. Some examples show the generality of this type of measure:

Example 4.1. Let $F(x) = x$, then the Stieltjes measure is nothing but the Lebesgue measure on the real line, that is, the extension of the concept of length of an interval.

Example 4.2. Let F be a jump function with discontinuity points $\{x_1, x_2, \ldots\}$ and corresponding jumps $\{h_1, h_2, \ldots\}$. The measure is of course:

$$m(\{x_n\}) = h_n \qquad m\left(\{x_1, x_2, \ldots\}^c\right) = 0$$

Then every subset of $[\alpha, \beta)$ is μ_F-measurable because their measure depends only on countable points. Any set A has measure given by:

$$\mu_F(A) = \sum_{x_n \in A} h_n$$

This number exists by assumption. A Stieltjes measure generated by a jump function is called a discrete measure. All discrete random variables have CDF that are jump functions.

Example 4.3. Let F be an absolutely continuous non-decreasing function on $[\alpha, \beta)$. Absolutely continuous functions have a finite derivative a.e. let this derivative be $f = F'$. Then the Stieltjes measure μ_F is defined for all Lebesgue measurable sets and:

$$\mu_F(A) = \int_A f(x)\, dx$$

clearly in this case $\mu_F(\{x\}) = 0$ because $\{x\}$ has Lebesgue measure 0.

The result follows from Lebesgue theorem:

Theorem 4.1. ***(Lebesgue)*** *If F is absolutely continuous on $[a, b]$ then the derivative F' is integrable on $[a, b]$ and:*

$$F(\beta) - F(\alpha) = \int_\alpha^\beta F'(x)\, dx$$

Proof. Kolmogorov and Fomin (2012, Sec. 33, Thm. 6).

□

Applying this theorem here we get:

$$m(\alpha, \beta) = m[\alpha, \beta] = m(\alpha, \beta] = m[\alpha, \beta) = \int_{\alpha}^{\beta} f(x)\, dx$$

Because f is non-negative and integrable wrt all Lebesgue-measurable subsets of $[a, b]$ $\left(\mathcal{B}_{[a,b]}\right)$ we know by proposition (3.3) that

$$\mu_F(A) = \int_A f(x)\, dx$$

is a measure on $\left([a, b], \mathcal{B}_{[a,b]}\right)$ that coincides with m, because the extension is unique we get that μ_F is the Stieltjes measure we are looking for.

This type of measure is called absolutely continuous and is related to continuous random variables.

Now we can define the integral with respect to a Stieltjes measure:

Definition 4.2. (Lebesgue-Stieltjes integral) Let μ_F be Stieltjes measure with generating function F, and let g be a μ_F-measurable function, then the integral is defined as:

$$\int_a^b g(x)\, dF(x) = \int_{[a,b]} g(x)\, d\mu_F$$

If μ_F is discrete with $F(x) = \sum_{x_n \leq x} h(x_n)$, then we have:

$$\int_a^b g(x)\, dF(x) = \sum_n g(x_n)\, h_n$$

If μ_F is absolutely continuous then:

$$\int_a^b g(x)\, dF(x) = \int_a^b g(x)\, f(x)\, dx$$

As hinted above in probability Stieltjes measures arise naturally. Let ξ be a random variable and define $F(x) = \Pr(\xi < x)$, then as noted above F is non-decreasing and continuous from the left, moreover $F(-\infty) = 0$ and $F(\infty) = 1$. The Lebesgue-Stieltjes

measure allows us to define the expected value and variance of the random variable as:

$$E[\xi] = \int_{-\infty}^{\infty} x dF(x) \qquad V[\xi] = \int_{-\infty}^{\infty} (x - E[\xi])^2 \, dF(x) \, .$$

These definitions are valid for discrete and continuous random variables.

5. Markov Processes

As seen in Section 6.2 a great deal of problems can be expressed in a recursive setting, and the use of recursive methods can provide solution to problems that would otherwise be impossible to handle. When dealing with random variables the same topic arises, in particular one can think of a sequence made by the realizations of a random variable, because the sequence is ordered one can also think of each element of the sequence being realized sequentially, in this way its natural to consider the case in which one element of the sequence depends on the value of the previous element. More formally, when the distribution of one element of the sequence depends on the realization of the previous element. Markov processes are processes that behave in this way.

The objective is to introduce shocks to a dynamic program, so we start by considering the deterministic dynamic program of Section 6.2, characterized by the Bellman equation:

$$v(x) = \sup_{y \in \Gamma(x)} \left\{ F(x, y) + \beta v(y) \right\} \tag{5.1}$$

The idea is to add a random variable whose realization z will affect the problem, z is a state of the problem and its drawn each period from a distribution characterized by the measure λ. Formally consider $(Z, \mathcal{Z}, \lambda)$ a probability space, then we can define the problem to be::

$$v(x, z) = \sup_{y \in \Gamma(x)} \left\{ F(x, y, z) + \beta \int v\left(y, z'\right) \lambda\left(dz'\right) \right\} \tag{5.2}$$

Recall that $\lambda : \mathcal{Z} \to \mathbb{R}_+$ maps sets of the σ-algebra $\mathcal{Z}$ to real numbers. The problem above can be solved using the results of Sections (1) to (4), but it is not general enough for our purposes because the distribution of z is fixed, and each draw is taken (each period) from the same distribution.

In general we want the distribution of z' to be influenced by the previous draw z, for this we need a special type of function, $Q : Z \times \mathcal{Z} \to \mathbb{R}$, such that for all $z \in Z$ it holds that $Q(z, \cdot)$ is a probability distribution for z'. This is called a transition function and it allows to express the problem as:

$$v(x, z) = \sup_{y \in \Gamma(x)} \left\{ F(x, y, z) + \beta \int v\left(y, z'\right) Q\left(z, dz'\right) \right\} \tag{5.3}$$

The objective is now to characterize transition functions and the properties of the process that they generate.

5.1. Transition functions

Definition 5.1. (Transition Function) Let $(Z, \mathcal{Z})$ be a measurable space. A transition function is a function $Q : Z \times \mathcal{Z} \to [0, 1]$ such that:

(a) For each $z \in Z$ the function $Q(z, \cdot)$ is a probability measure on $(Z, \mathcal{Z})$.

(b) For each $A \in \mathcal{Z}$ the function $Q(\cdot, A)$ is a $\mathcal{Z}$-measurable function.

The interpretation is that for all current value of the random variable the transition function induces a probability measure for next period's value of the variable. Then $Q(a, A)$ is the probability that $z' \in A$ if the current value of the variable is a.

$$Q(a, A) = \Pr\left(z' \in A | z = a\right)$$

Any transition function defines two operators that will be of great importance later.

Definition 5.2. Let Q be a transition function on a measurable space $(Z, \mathcal{Z})$. Define $\mathcal{F}$ as the set of $\mathcal{Z}$-measurable functions and Λ the set of probability measures on $(Z, \mathcal{Z})$.

(a) The Markov operator of Q is an operator T defined on the set of $\mathcal{Z}$-measurable functions:

$$Tf(z) = \int f\left(z'\right) Q\left(z, dz'\right)$$

for all $z \in Z$. T is the expected value of f in the next period if today's realization is z.

(b) The Adjoint operator of Q is $T^\star$ is an operator defined on probability measures on $(Z, \mathcal{Z})$:

$$T^\star \lambda(A) = \int Q(z, A) \lambda(dz)$$

for all $A \in \mathcal{Z}$. T gives the probability that $z' \in A$ if the current value of z is drawn from probability distribution λ.

These operators are important because they will allow to characterize the distribution of a sequence of random variables starting at some initial distribution. This is the objective when solving a stochastic dynamic programming problem. In order for T and $T^\star$ to be useful it is first necessary to check that they are sufficiently well behaved. The following propositions will establish that the operators can be used recursively and their proof will be instructive of how proofs go in measure theory.

Proposition 5.1. *Let $(Z, \mathcal{Z})$ be a measurable space and Q a transition function on that space with Markov operator T. Then $T : M^+(Z, \mathcal{Z}) \to M^+(Z, \mathcal{Z})$ where $M^+(Z, \mathcal{Z})$ is the space of nonnegative, extended variable $\mathcal{Z}$-measurable functions.*

Proof. The proof is done iteratively, first for indicator functions, then it is generalized to simple functions and then to arbitrary nonnegative measurable functions.

First, for any $f \in M^+$ we have that Tf is a nonnegative function of extended real value, this follows immediately, then it is left to check that Tf is also measurable.

Case 1. Let $A \in \mathcal{Z}$ and $f = \chi_A$, where χ_A is by construction measurable and nonnegative. Then:

$$Tf(z) = \int f\left(z'\right) Q\left(z, dz'\right) = \int \chi_A\left(z'\right) Q\left(z, dz'\right) = \int_A Q\left(z, dz'\right) = Q(z, A)$$

Because $Q(\cdot, A)$ is measurable (as a function of z for fixed A) by definition we establish measurability of Tf.

Case 2. Let f be a simple function then there exists (finitely many) indicator functions such that: $f(z) = \sum_{i=1}^{n} a_i \chi_{A_i}(z)$. Then the Markov operator gives:

$$\begin{aligned} Tf(z) &= \int f\left(z'\right) Q\left(z, dz'\right) \\ &= \int \sum_{i=1}^{n} a_i \chi_{A_i}\left(z'\right) Q\left(z, dz'\right) \\ &= \sum_{i=1}^{n} a_i \int_A \chi_{A_i}\left(z'\right) Q\left(z, dz'\right) \\ &= \sum_{i=1}^{n} a_i T\chi_{A_i} \end{aligned}$$

By the previous case each $T\chi_{A_i}$ is measurable. Then, Tf is also measurable because the sum and scalar product of measurable functions is also measurable .

Case 3. Let f be an arbitrary nonnegative, extended real value, measurable function. By proposition 2.5 we know that f can be expressed as the limit of point-wise convergent sequence of simple functions because it is measurable and nonnegative. So for all z we have:

$$\begin{aligned} Tf(z) &= \int f\left(z'\right) Q\left(z, dz'\right) \\ &= \int \lim \phi_n\left(z'\right) Q\left(z, dz'\right) \\ &= \lim \int \phi_n\left(z'\right) Q\left(z, dz'\right) \\ &= \lim T\phi_n(z) \end{aligned}$$

where the third step of interchanging the limit and the integral follows from Lebesgue's Monotone Convergence theorem.[2] Finally the pointwise limit of measurable functions is a measurable function (Proposition 2.4). The result follows because $T\phi_n$ is measurable by the previous case.

□

Corollary 5.1. *Let $(Z, \mathcal{Z})$ be a measurable space and Q a transition function on that space with Markov operator T. Then $T : B(Z, \mathcal{Z}) \to B(Z, \mathcal{Z})$ where $B(Z, \mathcal{Z})$ is the space of bounded $\mathcal{Z}$-measurable functions.*

Proof. Let f be a bounded measurable function, then if $0 \leq f \leq m$ it holds that $0 \leq Tf \leq m$, because $Q(z, \cdot)$ is a probability measure. Then Tf is bounded. Measurability follows from the proposition above by applying it to $f = f^+ - f^-$.

□

This allows us to apply iteratively the operator to a function because if $f \in B(Z, \mathcal{Z})$ then $Tf \in B(Z, \mathcal{Z})$, which allows to evaluate $T(Tf)$, and so on. It will also be important to apply the adjoint operator iteratively to a probability measure. The following proposition will enable us to do so.

Proposition 5.2. *Let $(Z, \mathcal{Z})$ be a measurable space and Q a transition function on that space with Adjoint operator $T^\star$. Then $T^\star : \Lambda(Z, \mathcal{Z}) \to \Lambda(Z, \mathcal{Z})$ where $\Lambda(Z, \mathcal{Z})$ is the space of probability measures on $(Z, \mathcal{Z})$.*

Proof. Let $\lambda \in \Lambda(Z, \mathcal{Z})$ and consider $T^\star\lambda(A) = \int Q(z, A)\, d\lambda(dz)$.

(a) $T^\star\lambda \geq 0$ for all (z, A) because $Q(z, A) \geq 0$.

(b) $T^\star\lambda(\emptyset) = \int Q(z, \emptyset)\lambda(dz) = \int 0\lambda(dz) = 0$, because $Q(z, \cdot)$ is a probability measure.

(c) $T^\star\lambda(Z) = \int Q(z, Z)\lambda(dz) = \int 1\lambda(dz) = 1$, because $Q(z, \cdot)$ is a probability measure.

(d) It is left to show that $T^\star\lambda$ is countably additive. Let $\{A_i\} \subseteq \mathcal{Z}$ be a sequence of disjoint sets and $A = \cup A_i$, then:

$$\sum_{i=1}^{\infty} T^\star\lambda(A_i) = \sum_{i=1}^{\infty} \int Q(z, A_i)\lambda(dz_i) = \int \left(\sum_{i=1}^{\infty} Q(z, A_i) \right) \lambda(dz_i) = \int Q(z, A)\lambda(dz_i) = T^\star\lambda(A)$$

where $\sum Q(z, A_i) = Q(z, A)$ follows from Q being a σ-additive measure and interchange of the sum and the integral can be done because of the Lebesgue's monotone convergence theorem.

[2] The theorem states that if $\{f_n\}$ is a monotone increasing sequence of nonnegative measurable functions then that converges pointwise to f then $\int f d\mu = \lim \int f_n d\mu$. Recall from proposition 2.5 that the sequence $\{\phi_n\}$ of simple functions can be chosen to be monotone increasing.

□

The following result establishes a duality between the Markov operator and its adjoint, in words it says that the expected value of a function tomorrow can be computed with either operator.

Proposition 5.3. *Let $(Z, \mathcal{Z})$ be a measurable space and Q a transition function on that space. Then for any function $f \in B(Z, \mathcal{Z})$ (or more generally $f \in M^{+}(Z, \mathcal{Z})$) it holds that:*

$$\int (Tf(z))\,\lambda(dz) = \int f\left(z'\right) T^{\star}\lambda\left(dz'\right) = \int\int f\left(z'\right) Q\left(z, dz'\right)\lambda(dz)$$

Then to obtain the expected value of function f tomorrow given a distribution λ of z today the order of integration does not matter.

Proof. Stokey, Lucas, and Prescott (1989, Sec. 8.1).

□

We can now define a sequence of probability measures over the sequence of random variables by iterating over Q with the Markov operator.

$$\begin{aligned} Q^{1}(z, A) &= Q(z, A) \\ &\vdots \\ Q^{n+1}(z, A) &= \left(TQ^{n}(\cdot, A)\right)(z) = \int Q^{n}\left(z', A\right) Q\left(z, dz'\right) \end{aligned}$$

Then if a shock is drawn sequentially from Q the function $Q^{n}(z, A)$ will give the probability of going from initial point z to a value in set A in exactly n periods. Its easy to show that each Q^{n} is a transition function.

Finally its clear that starting from an initial probability (λ) the Adjoint operator can be used to define a sequence of probability measures $\left\{\lambda^{n}\right\}$ as $\lambda^{n} = T^{\star}\lambda^{n-1}$, we interpret λ as the distribution of the state z in the initial period and λ_{n} the (unconditional) distribution of z in the n^{th} period.

As a side note a transition function can have stronger properties that are of great use in stochastic dynamic programming:

Definition 5.3. (Feller Property) A transition function Q has the feller property if its Markov operator maps the set of continuous bounded function into itself. $T : C(Z) \to C(Z)$.

Definition 5.4. (Monotone transition functions) A transition function Q is said to be monotone if its Markov operator maps nondecreasing functions to nondecreasing functions.

5.2. Probability measures on spaces of sequences

The idea now is to study sequences of random variables and their probability distributions, this can be done using the transition function defined above.

The first task at hand is to define a probability distribution on a finite sequence of variables. For this let $(Z, \mathcal{Z})$ be a measurable space and for $t < \infty$ let $\left(Z^t, \mathcal{Z}^t\right) = (Z \times \ldots \times Z, \mathcal{Z} \times \ldots \times \mathcal{Z})$ be a product space. Now let Q be a transition function on $(Z, \mathcal{Z})$. A probability measure on the sequence given z_0, the initial value of the variable is:

Definition 5.5. (Probability measure on finite sequence) $\mu^t : Z \times \mathcal{Z}^t \to [0, 1]$ is the probability distribution for the finite sequence and its defined as:

$$\mu^t(z_0, B) = \int_{A_1} \ldots \int_{A_t} Q(z_{t-1}, dz_t) \cdot Q(z_{t-2}, dz_{t-1}) \cdots Q(z_0, dz_1)$$

where $B = A_1 \times \ldots \times A_t \in \mathcal{Z}^t$ is a rectangle in $\mathcal{Z}^t$. It can be shown that it is sufficient to define μ^t only for this type of set, because it can then be extended uniquely to measurable sets on $\mathcal{Z}^t$ by the Caratheodory and Hahn extension theorems.

The next task is to handle infinite sequences of realizations of z. To do this we need to be able to induce a σ-algebra $\mathcal{Z}^\infty$ on the set of infinite sequences and then a probability measure on that σ-algebra.

To do this define the set of finite-measurable rectangles. These sets establish outcomes for the variables for the first T periods, leaving unspecified what happens to the sequence afterwards.

Definition 5.6. (Finite-Measurable Rectangles) B is a finite measurable rectangle if its of the form:

$$B = A_1 \times \ldots \times A_T \times Z \times Z \times \ldots$$

for some finite T. Let $\mathcal{C}$ be the set of all finite measurable rectangles. Let $\mathcal{A}^\infty$ be the set of all finite unions of set in $\mathcal{C}$.

It can be shown that $\mathcal{A}^\infty$ is an algebra, then one can define $\mathcal{Z}^\infty$ to be the σ-algebra induced by $\mathcal{A}^\infty$. Then one can define a measure on finite-measurable rectangles $\mathcal{C}$ just as before, extend it to the algebra $\mathcal{A}^\infty$, and the extend the extension to $\mathcal{Z}^\infty$. This proves the existence of a measure for infinite sequences that coincides with our notion of measure for finite-measurable rectangles.

Now we can define what a stochastic process is:

Definition 5.7. (Stochastic Process) Let $(\Omega, \mathcal{F}, P)$ be a probability space. A stochastic process on $(\Omega, \mathcal{F}, P)$ is an increasing sequence of σ-algebras $\mathcal{F}_1 \subseteq \mathcal{F}_2 \subseteq \ldots \subseteq \mathcal{F}$, a measurable space $(Z, \mathcal{Z})$and a sequence of functions $\sigma_t : \Omega \to Z$ such that each σ_t is $\mathcal{F}_t$-measurable.

Definition 5.8. (Sample Path) Let $\omega \in \Omega$, then $(\sigma_1(\omega), \sigma_2(\omega), \ldots)$ is called the sample path of the stochastic process given ω.

Each σ_t is just a random variable that takes a value on Z given some event on Ω. In almost all cases we will have $(Z, \mathcal{Z}) = (\mathbb{R}, \mathcal{B})$. The selection of the probability space $(\Omega, \mathcal{F}, P)$ is also standard. Because we are interested in the behavior of infinite sequences of the realizations of the random variable we can set $(\Omega, \mathcal{F}, P) = (Z^\infty, \mathcal{Z}^\infty, \mu(z_0, \cdot))$. The restriction that the σ-algebras are increasing follows from the draws being taken sequentially, this σ-algebras will be interpreted as possible histories, and any future history must include all of the possible previous histories from which it could have followed.

Given a stochastic process we can use probability measure P to induce measures on finite sets of sample paths.

Definition 5.9. (Probabilities on Paths) Let $C \in \mathcal{Z}^n$ we can define:

$$P_{t+1,\ldots,t+n}(C) = P\left(\left\{\omega \in \Omega \mid (\sigma_{t+1}(\omega), \ldots \sigma_{t+n}(\omega)) \in C\right\}\right)$$

This is the probability that an event occurs and the sample path lies in C between periods $t+1$ and $t+n$.

Definition 5.10. (Stationary Stochastic Process) A stochastic process is said to be stationary if $P_{t+1,\ldots,t+n}(C)$ is independent of t for all n and C. That is, if it does not matter the point in time where we start the sequence.

Definition 5.11. (Conditional probability) Let $P_{t+1,\ldots,t+n}(C|a_{t-s}, \ldots, a_{t-1}, a_t)$ be the conditional probability of the event $\left\{\omega \in \Omega \mid (\sigma_{t+1}(\omega), \ldots \sigma_{t+n}(\omega)) \in C\right\}$ given that the event $\left\{\omega \in \Omega \mid \sigma_\tau(\omega) = a_\tau\right\}$ happened.

Now we can define what a Markov process is:

Definition 5.12. (Markov Process) A stochastic process is a Markov process if:

$$P_{t+1,\ldots,t+n}(C|a_{t-s}, \ldots, a_t) = P_{t+1,\ldots t+n}(C|a_t)$$

for $t = 1, 2, \ldots$, $n = 1, 2, \ldots$, $s = 1, 2, \ldots, t-1$ and $C \in \mathcal{Z}^n$.

The distribution of the path of a Markov process only depends on the last realization.

A general setting is easy to construct using a transition function Q. Let $(\Omega, \mathcal{F}, P) = (Z^\infty, \mathcal{Z}^\infty, \mu(z_0, \cdot))$ and for each T define $\mathcal{A}^T$ as the collection of all finite-measurable sets:

$$B = A_1 \times \ldots \times A_T \times Z \times Z \times \ldots$$

As before this forms an algebra, let $\mathcal{F}^T$ be the σ-algebra generated by $\mathcal{A}^T$. Clearly $\mathcal{F}^t \subseteq \mathcal{F}^{t+1}$. Then we can define the sequence of functions $\tilde{z}_t : \Omega \to Z$ as:

$$\tilde{z}_t(\omega) = \tilde{z}_t(a_1, a_2, \ldots) = a_t$$

so that it selects the t^{th} realization of the sequence ω. These functions are clearly $\mathcal{F}^t$ measurable, because they don't contain information about future realizations of the variable.

The definition of P through Q can be used to verify that this process is a Markov process. Moreover it holds that:

$$P_{t+1}(C|a_{t-s}, \ldots, a_t) = P_{t+1}(C|a_t) = Q(a_t, C)$$

for $C \in \mathcal{Z}$.

5.3. Markov chains

We now zoom into a special type of Markov process that is particularly useful in applications of dynamic programming. A Markov chain (or finite state Markov chain):

Definition 5.13. (Markov Chain) A Markov chain is a Markov process defined on a space $Z = \{z_1, \ldots, z_l\}$ with finite dimension (finitely many elements).

The relevance of Markov chains resides in two observations. First, they allow for a simple characterization of their transition function, as shown below. Second, most computational methods (and thus applications) of dynamic programming discretize the state space, effectively imposing that the space Z is finite.

Before characterizing the transition function of a Markov chain it is useful to recall that the natural σ-algebra over Z is $\mathcal{Z} = 2^Z$ (the power set), and that the space of probabilities distributions over Z is formed by vectors $p \in \mathbb{R}^l$ such that $p_i \geq 0$ and $\sum_{i=1}^{l} p_i = 1$[3]. The

[3]Formally $p \in \Delta^l$, where $\Delta^l = \left\{ p \in \mathbb{R}^l_+ | \sum_{i=1}^{l} p_i = 1 \right\}$ is the $l-1$ dimensional simplex. This same set is particularly useful in characterizing price systems in finite dimensional exchange economies.

transition function of the Markov process can be then characterized by a matrix:

Definition 5.14. (Markov Matrix / Stochastic Matrix) A square matrix $\Pi = \left[\pi_{ij}\right]$ of dimensions $l \times l$ is considered a Markov (or stochastic) matrix if $\pi_{ij} \geq 0$ for all i and j, and $\sum_{j=1}^{l} \pi_{ij} = 1$ for all i. Equivalently, if its rows are probability distributions on Z: $\pi_i \in \Delta^l$.

The transition function of a Markov chain is a function $Q : Z \times \mathcal{Z} \to \mathbb{R}_+$ that gives the probability of a given set $A \in \mathcal{Z}$ given a current state z_i. We can then construct a Markov matrix by setting:

$$\pi_{ij} = Q\left(z_i, \left\{z_j\right\}\right) = \Pr\left(z_{t+1} = z_j \,|z_t = z_i\right)$$

So π_{ij} is interpreted as the probability that $z_{t+1} = z_j$ conditional on $z_t = z_i$. The row $\pi_i = \left(\pi_{i1}, \ldots, \pi_{il}\right)$ is the conditional probability of z_{t+1}, given that $z_t = z_i$.

We can also go the other way, constructing a transition function Q from a Markov matrix Π. Let $A \in \mathcal{Z}$, because Z is finite we can define $A = \left\{z_{a_1}, \ldots, z_{a_n}\right\}$ where $a_1, \ldots, a_n$ are $n \leq l$ indices. This gives:

$$Q\left(z_i, A\right) = \sum_{j=1}^{n} \pi_{ia_j}$$

We can now define the Markov operator and the adjoint Markov operator of a Markov chain making use of the Markov matrix (Markov!).

(a) Recall that The Markov operator of Q is an operator T defined on the set of $\mathcal{Z}$-measurable functions:

$$Tf(z) = \int f\left(z'\right) Q\left(z, dz'\right)$$

for all $z \in Z$. T is the expected value of f in the next period if today's realization is z. For Markov chains the function f can be reduced to a row-vector $\vec{f} = \left(f(z_1), \ldots, f\left(z_l\right)\right) \in \mathbb{R}^l$, which reduces the integral to:

$$Tf\left(z_i\right) = \vec{f}\pi_i'$$

more generally we have:

$$Tf = \vec{f}\Pi'$$

the i^{th} element of Tf (which is an l-dimensional vector) corresponds to: $E\left[f\left(z_{t+1}\right) | z_t = z_i\right]$.

(b) Recall that the adjoint operator of Q is $T^\star$ is an operator defined on probability measures

on $(Z, \mathcal{Z})$:

$$T^{\star}\lambda(A) = \int Q(z, A)\,\lambda(dz)$$

for all $A \in \mathcal{Z}$. T gives the probability that $z^{'} \in A$ if the current value of z is drawn from probability distribution λ. Because the space is finite we can represent probabilities distributions as vectors in Δ^{l}. Let $p \in \Delta^{l}$ be an initial distribution on Z, we want to know the distribution on Z for the next period ($\hat{p}$):

$$T^{\star}\,p = \hat{p} = p\Pi \qquad \text{where: } \hat{p}_j = \sum_{i=1}^{l} p_i \pi_{ij}$$

The j^{th} element of $T^{\star}\,p$ (which is an l-dimensional row-vector) corresponds to the unconditional probability that $z_{t+1} = z_j$: $\Pr\left(z_{t+1} = z_j\right)$.

It shouldn't be a surprise that the Markov operator is characterized by $\Pi^{'}$ and the adjoint operator by its transpose Π.[4]

As with general Markov processes there is a special interest in the limit behavior of the adjoint operator $\left(\lim_{n\to\infty} \Pi^{n} p\right)$, in particular the existence and properties of an invariant distribution, that is $p^{\star}$ such that $p^{\star} = p^{\star}\Pi$ (generally $\lambda^{\star} = T\lambda^{\star}$). The problem of finding an invariant distribution is frequently cast as an eigenvector problem. $p^{\star}$ is the eigenvector associated with any unit-eigenvalue of Π.

Another property that will be of interest is the presence of Ergodic sets. These are subsets of the space $E \subseteq Z$ that the process never leaves once it takes a value in them. Formally:

Definition 5.15. (Ergodic Set) A set $E \subseteq Z$ is ergodic if and only if $Q\left(z_i, E\right) = 1$ for all $z_i \in E$ and there does not exist a proper subset $E^{'} \subset E$ that is ergodic.

The ergodic sets are important because they tell us sections of the state space that are of interest. Only ergodic sets have positive mass in the invariant distribution.

Following SLP we now show 5 examples of the possible limit behavior of Markov chains. After the examples we state the main results on the existence and uniqueness of ergodic sets, invariant distributions, and the convergence of the sequences $\left\{\frac{1}{n}\sum_{k=0}^{n}\Pi^{k}\right\}$ and $\left\{\Pi^{n}\right\}$. (clearly if the second sequence converges so does the first one).

[4] If vectors are assumed to be columns instead of rows then $Tf = \Pi f$ and $T^{\star}\,p = \Pi^{'}\,p$. The adjoint is characterized as the transpose of the Markov operator in any case.

Example 5.1. (Uniqueness of ergodic set, convergence of $\{\Pi^n\}$) Let $l = 2$ and consider the Markov matrix:

$$\Pi = \begin{bmatrix} 3/4 & 1/4 \\ 1/4 & 3/4 \end{bmatrix}$$

Clearly the only ergodic set is Z because one has positive probability of going to z_1 or z_2 starting in any state. Moreover:

$$\lim_{n\to\infty} \Pi^n = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix}$$

The invariant distribution is then $p^\star = (1/2, 1/2)$. Moreover $\lim \left(p_0 \Pi^n \right) = p^\star$ for all $p_0 \in \Delta^2$.

Note: Convergence is easily defined in this setup because the limit is taken element wise.

Example 5.2. (Uniqueness of a ergodic set, convergence of $\{\Pi^n\}$) Let $l = 3$ and $\gamma \in (0, 1)$. Consider the Markov matrix:

$$\Pi = \begin{bmatrix} 1-\gamma & \gamma/2 & \gamma/2 \\ 0 & 1/2 & 1/2 \\ 0 & 1/2 & 1/2 \end{bmatrix}$$

There is a unique ergodic set $E = \{z_2, z_3\} \neq Z$. The state z_1 is never reached again once you leave it. One can also show:

$$\Pi^n = \begin{bmatrix} (1-\gamma)^n & (1-(1-\gamma)^n)/2 & (1-(1-\gamma)^n)/2 \\ 0 & 1/2 & 1/2 \\ 0 & 1/2 & 1/2 \end{bmatrix}$$

clearly $\{\Pi^n\}$ converges:

$$\lim_{n\to\infty} \Pi^n = \begin{bmatrix} 0 & 1/2 & 1/2 \\ 0 & 1/2 & 1/2 \\ 0 & 1/2 & 1/2 \end{bmatrix}$$

and the invariant distribution is $p^\star = (0, 1/2, 1/2)$.

Example 5.3. (Cyclical sets , convergence of $\left\{ \frac{1}{n} \sum_{k=0}^{n} \Pi^k \right\}$) Consider an l-dimensional Markov chain and order its states into two subsets, the first one with k elements and the

second one with $l - k$ elements. Suppose the Markov matrix has the form:

$$\Pi = \begin{bmatrix} 0 & \Pi_1 \\ \Pi_2 & 0 \end{bmatrix}$$

where the first matrix Π_1 is of dimension $k \times (l - k)$ and matrix Π_2 of dimension $(l - k) \times k$. Clearly if at one period one is in the first subset the next period one will be in the second subset, and vice-versa. So there are no proper subsets that form an ergodic subset, instead the process cycles from the first subset to the second period by period.

$$\Pi^{2n} = \begin{bmatrix} (\Pi_1\Pi_2)^n & 0 \\ 0 & (\Pi_2\Pi_1)^n \end{bmatrix} \qquad \Pi^{2n+1} = \begin{bmatrix} 0 & (\Pi_1\Pi_2)^n \Pi_1 \\ (\Pi_2\Pi_1)^n \Pi_2 & 0 \end{bmatrix}$$

In this example the sequence $\left\{\Pi^n\right\}$ does not converge but its odd and even elements do, then the sequence $\left\{\frac{1}{n}\sum_{k=0}^{n}\Pi^k\right\}$ does converge.

For example if $l = 4$, $k = 2$ and $\Pi_1 = \Pi_2$:

$$\lim_{n\to\infty} \Pi^{2n} = \begin{bmatrix} 1/2 & 1/2 & 0 & 0 \\ 1/2 & 1/2 & 0 & 0 \\ 0 & 0 & 1/2 & 1/2 \\ 0 & 0 & 1/2 & 1/2 \end{bmatrix} \qquad \lim_{n\to\infty} \Pi^{2n+1} = \begin{bmatrix} 0 & 0 & 1/2 & 1/2 \\ 0 & 0 & 1/2 & 1/2 \\ 1/2 & 1/2 & 0 & 0 \\ 1/2 & 1/2 & 0 & 0 \end{bmatrix}$$

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n}\Pi^k = \begin{bmatrix} 1/4 & 1/4 & 1/4 & 1/4 \\ 1/4 & 1/4 & 1/4 & 1/4 \\ 1/4 & 1/4 & 1/4 & 1/4 \\ 1/4 & 1/4 & 1/4 & 1/4 \end{bmatrix}$$

An invariant distribution is found as one of the rows of the last limit: $p^\star = (1/4, 1/4, 1/4, 1/4)$.

Example 5.4. (Two ergodic sets, Infinitely many invariant distributions) Consider an l-dimensional Markov chain and order its states into two subsets, the first one with k elements and the second one with $l - k$ elements. Suppose the Markov matrix has the form:

$$\Pi = \begin{bmatrix} \Pi_1 & 0 \\ 0 & \Pi_2 \end{bmatrix}$$

where the first matrix Π_1 is of dimension $k \times (l - k)$ and matrix Π_2 of dimension $(l - k) \times k$.

Clearly once the process enters the first subset it never leaves it. The same goes for the second subset. Then they are both ergodic. Also $\Pi^n = \begin{bmatrix} \Pi_1^n & 0 \\ 0 & \Pi_2^n \end{bmatrix}$, this sequence converges if and only if $\{\Pi_1^n\}$ and $\{\Pi_2^n\}$ converge. Let $l = 4$, $k = 2$ and $\Pi_1 = \Pi_2 = \begin{bmatrix} 3/4 & 1/4 \\ 1/4 & 3/4 \end{bmatrix}$, then:

$$\lim_{n\to\infty} \Pi^n = \begin{bmatrix} 1/2 & 1/2 & 0 & 0 \\ 1/2 & 1/2 & 0 & 0 \\ 0 & 0 & 1/2 & 1/2 \\ 0 & 0 & 1/2 & 1/2 \end{bmatrix}$$

There are two invariant distributions: $p_1^\star = (1/2, 1/2, 0, 0)$ and $p_2^\star = (0, 0, 1/2, 1/2)$. But any convex combination of them is also an invariant distribution.

Example 5.5. (Two ergodic sets, Infinitely many invariant distributions) Let $l = 3$, $\gamma \in (0, 2)$ and $\alpha, \beta \geq 0$ such that $\alpha + \beta = 1$. Consider the Markov matrix:

$$\Pi = \begin{bmatrix} 1-\gamma & \gamma\alpha & \gamma\beta \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

As in the second example s_1 is a transient state (once you leave it you never come back), but there are now two ergodic sets $\{s_2\}$ and $\{s_3\}$. We also have:

$$\lim_{n\to\infty} \Pi^n = \lim_{n\to\infty} \begin{bmatrix} (1-\gamma)^n & \left(1-(1-\gamma)^n\right)\alpha & \left(1-(1-\gamma)^n\right)\beta \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & \alpha & \beta \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The sequence $\{\Pi^n\}$ converges and there are two invariant distributions $p_1^\star = (0, 1, 0)$ and $p_2^\star = (0, 0, 1)$ given by the second and third rows of the limiting matrix. The first row is a convex combination of the limiting distribution.

Now we turn to the general results. The following theorem encompasses all the possible outcomes of a Markov chain. In particular, an ergodic set and a limit distribution always exist, but they need not be unique, and although the sequence $\{\Pi^n\}$ need not converge, the sequence $\left\{\frac{1}{n}\sum_{k=0}^{n}\Pi^k\right\}$ always converges, and its limit gives away the invariant distributions.

Theorem 5.1. *Let* $Z = (z_1, \ldots, z_l)$ *and denote the stochastic matrix by its elements:* $\Pi = \left[\pi_{ij}\right]$. *The powers of* Π *are also denoted by its elements* $\Pi^n = \left[\pi_{ij}^{(n)}\right]$.

(a) *Z can be partitioned into* $M \geq 1$ *ergodic sets and a transient set (an ergodic set always exists).*

(b) *The sequence* $\left\{\frac{1}{n}\sum_{k=0}^{n}\Pi^k\right\}$ *always converges to a Markov matrix* $\overline{\Pi}$.

(i) *For any* $p_0 \in \Delta^l$ *and* $p_k = p_0\Pi^k$ *it holds that:* $\frac{1}{n}\sum_{k=0}^{n} p_k \to p_0\overline{\Pi}$.

(c) *Each row of* $\overline{\Pi}$ *is an invariant distribution, and every invariant distribution is a convex combination of the rows of* $\overline{\Pi}$ *(so* $p_0\overline{\Pi}$ *is an invariant distribution for all* $p_0 \in \Delta^l$*).*

We can strengthen these results by imposing extra structure on Π. We can get uniqueness of the ergodic set and the invariant distribution under a "reachability" condition (at least one state should be reachable in finite time starting from anywhere).

Theorem 5.2. *Let Z and* Π *as in Theorem 5.1.* Π *has a unique ergodic set if and only if there exists a state* z_j *such that for all* $i \in (1, \ldots, l)$ *there exist* $n \geq 1$ *such that* $\pi_{ij}^{(n)} > 0$.

Moreover, if this is the case Π *has a unique invariant distribution* $p^\star$ *and all rows of* $\overline{\Pi}$ *are equal to* $p^\star$ *(so for any* $p_0 \in \Delta^l$ *we have* $p_0\overline{\Pi} = p^\star$*).*

The previous result still does not rule out cyclicality in the ergodic set. We can get this under a "mixing" condition.

Theorem 5.3. *Let Z and* Π *as in Theorem 5.1. For* $n = 1, 2, \ldots$ *and* $j = 1, \ldots, l$ *define* $\epsilon_j^{(n)} = \min_i \pi_{ij}^{(n)}$ *and* $\epsilon^{(n)} = \sum_{j=1}^{l} \epsilon_j^{(n)}$. *Z has a unique ergodic set without cyclically moving subsets if and only if for some* $N \geq 1$ *it holds that* $\epsilon^{(N)} > 0$.

Moreover, if this is the case Π *has a unique invariant distribution* $p^\star$ *and the sequence* $\{\Pi^n\}$ *converges (so for any* $p_0 \in \Delta^l$ *we have* $\lim p_0\Pi^n = p^\star$*).*

In this notation, $\pi_{ij}^{(n)}$ is the probability that state j is reached from state i in n steps. $\epsilon_j^{(n)}$ measures the lowest such probability, because we don't know from which state we start from we need to know that the condition is satisfied for all states. This mixing property is stronger than the first one because we need there to be at least one column of non-zero elements, guaranteeing mixing towards one state (j) starting from any state (i), uniformly in time (the same time (N) for all the initial states).

5.4. Weak convergence of monotone Markov processes

In the previous section we obtained results characterizing the limiting behavior of Markov processes when the state space is finite. Unfortunately these results do not extend immediately to general Markov processes. The problem at hand is to establish when the sequence of distribution functions $\{\lambda_n\}$, constructed as $\lambda_n = T^\star \lambda_{n-1}$ with λ_0 given, converges. We must first define what it means for a sequence of distributions to converge. The simplest definition comes in the form of set-wise convergence (the equivalent of point-wise convergence for functions):

Definition 5.16. (Set-wise Convergence) Let $(Z, \mathcal{Z})$ be a measurable space and $\Lambda(Z, \mathcal{Z})$ the set of probability distributions. Consider a sequence $\{\lambda_n\} \subseteq \Lambda(Z, \mathcal{Z})$, we say that $\lambda_n \overset{s.w}{\to} \lambda \in \Lambda(Z, \mathcal{Z})$ if $\lambda_n(A) \to \lambda(A)$ for all $A \in \mathcal{Z}$.

This notion of convergence is intuitive but it turns out to be too strong for most applications. The following proposition shows why:

Definition 5.17. Let $(Z, \mathcal{Z})$ be a measurable space and $\Lambda(Z, \mathcal{Z})$ the set of probability distributions. Consider a sequence $\{\lambda_n\} \subseteq \Lambda(Z, \mathcal{Z})$. $\{\lambda_n\}$ converges set-wise to λ if and only if $\lim \int f(z)\, d\lambda_n = \int f(z)\, d\lambda$ for all bounded and measurable functions $f \in B(Z, \mathcal{Z})$.

Thus asking for set-wise convergence requires the expected value of a large class of functions to converge. A way to weaken this is to limit the space of functions for which convergence is required.

Definition 5.18. (Weak Convergence) Let (Z, ρ) be a metric space and $\mathcal{Z}$ the Borel set of Z. Define $\Lambda(Z, \mathcal{Z})$ as the set of probability distributions. Consider a sequence $\{\lambda_n\} \subseteq \Lambda(Z, \mathcal{Z})$, we say that $\{\lambda_n\}$ converges weakly to $\lambda \in \Lambda(Z, \mathcal{Z})$ if $\lim \int f(z)\, d\lambda_n = \int f(z)\, d\lambda$ for all bounded and continuous functions $f \in C(Z)$.

The main results we will obtain establish the existence of an invariant distribution under a continuity assumption on the Markov operator (the Feller property). We can then ensure uniqueness if the Markov operator is monotone and a mixing condition is satisfied, along with uniqueness we will obtain the weak convergence of $\left\{T^{\star n}\lambda_0\right\}$.

In what follows we consider $Z \subseteq \mathbb{R}^l$ for $l < \infty$, with $\mathcal{Z}$ the Borel σ-algebra of Z. The Markov process is characterized by its transition function Q, its Markov operator $T : B(Z, \mathcal{Z}) \to B(Z, \mathcal{Z})$ and its adjoint operator $T^\star : \Lambda(Z, \mathcal{Z}) \to \Lambda(Z, \mathcal{Z})$. We also define the inner product $\langle f, \lambda \rangle = \int f(z)\, d\lambda$.

We first expand on the Feller property through the following proposition:

Proposition 5.4. *The following three statements are equivalent:*

(a) *(Feller property) If* $f \in C(Z)$ *then* $Tf \in C(Z)$.

(b) *If* $z_n \to z$ *then* $Q(z_n, \cdot) \to Q(z, \cdot)$ *(that is for all* $A \in \mathcal{Z}$*).*

(c) *If* $\lambda_n \to \lambda$ *then* $T^\star \lambda_n \to \lambda$

Then preserving continuity in conditional expected values (Tf is a conditional expected value) has equivalent statements in terms of the conditional distributions ($Q(z_n, \cdot)$) and unconditional distributions ($T^\star \lambda_n$). It turns out that continuity is enough to guarantee that an invariant distribution exists.

Theorem 5.4. *If* $Z \subseteq \mathbb{R}^l$ *is compact and* Q *satisfies the Feller property then an invariant distribution exists. That is, there is* $\lambda^\star \in \Lambda(Z, \mathcal{Z})$ *such that* $\lambda^\star = T^\star \lambda^\star$.

Yet, continuity is not enough to rule out the existence of many invariant distributions or of cycling sets. Monotonicity is needed for this. As before it is first useful to take a detour on what monotonicity (as in Definition 5.4) implies for distribution functions. We then have to impose an ordering of distribution functions to be able to talk about monotonicity.

Definition 5.19. (First Order Stochastic Dominance) A distribution μ (first order stochastically) dominates λ ($\mu \geq \lambda$) if $\int f(z)\, d\mu \geq \int f(z)\, d\lambda$ for all increasing, bounded and measurable function f.

In what follows we call a sequence $\{\lambda_n\}$ monotone if $\lambda_{n+1} \geq \lambda_n$ for all n, or if $\lambda_{n+1} \leq \lambda_n$ for all n. We can now establish the following result:

Proposition 5.5. *The following three statements are equivalent:*

(a) *(Monotone property) If* $f \in B(Z, \mathcal{Z})$ *is weakly increasing then* Tf *is also weakly increasing.*

(b) *Let* $\lambda, \mu \in \Lambda(Z, \mathcal{Z})$*. If* $\mu \geq \lambda$ *then* $T^\star \mu \geq T^\star \lambda$.

(c) *If* $z \geq z'$ *then* $Q(z, \cdot) \geq Q\left(z', \cdot\right)$ *(in the stochastic dominance sense)*

The last statement is particularly useful because it translates monotonicity of the Markov operator directly into monotonicity of the transition function ("better" states lead to "better" distributions).

Now we introduce the final condition needed for the main result of this section. It is a mixing condition akin to that in Theorems 5.2 and 5.3, along with a restriction on the form of the set Z. To see why it is necessary to go SLP exercises 12.12 and 12.13.

Assumption. *The set $Z = [a, b]$ is a closed and bounded rectangle in $\mathbb{R}^l$ characterized by a and b,*[5] *and there exists $z \in Z$, $\epsilon > 0$ and $N \geq 1$ such that:*

$$Q^N(a, [c, b]) \geq \epsilon \qquad Q^N(b, [a, c]) \geq \epsilon$$

Under this assumption it is possible to reach the "upper" region of the rectangle, $[c, b]$, in finite time starting from the "lower" corner (a), and it is possible to reach the "lower" region of the rectangle, $[a, c]$, in finite time starting from the "upper" corner (b). It is possible to show that if one can move through the set from the corners it is possible to do it from anywhere (under a monotonicity assumption).

Proposition 5.6. *Let Q satisfy monotonicity and the previous assumption for some tuple (c, ϵ, N), then:*

$$Q^N(z, [c, b]) \geq \epsilon \qquad Q^N(z, [a, c]) \geq \epsilon \qquad \text{for all } z \in Z$$

Finally we establish the convergence result.

Theorem 5.5. *Let $S = [a, b] \in \mathbb{R}^l$ be a rectangle and satisfy the assumption above. If Q is monotone and satisfies the Feller property, then Q has a unique invariant distribution $\lambda^\star$ and $T^{\star n}\lambda_0 \to \lambda^\star$ for all $\lambda_0 \in \Lambda(Z, \mathcal{Z})$.*

This completes the tools we need to tackle stochastic dynamic programming problems. In that note, it is interesting to note the similarities between Theorem 5.5 and the contraction mapping theorem and Blackwell's conditions used below to establish a unique solution for Bellman equations and the convergence to that solution from any starting point by iteratively applying the Bellman operator(see Section 6.1). While Theorem 5.5 does not establish a sense of distance (inherent in complete metric spaces) it does provide a very similar answer to the question of the existence and uniqueness of a fixed point characterizing the solution to a functional equation.

The stationary distribution of a Markov process solves the functional equation:

$$\lambda = T^\star \lambda.$$

The result in the theorem is a set of sufficient conditions for there to be a unique solution, $\lambda^\star$. These conditions have the Markov operator T preserve the continuity and boundedness of functions (recalling that the space of continuous and bounded functions is a complete metric space) and monotonicity (as in Blackwell's first sufficient condition for a contraction).

[5] A set $Z \subseteq \mathbb{R}^l$ is a closed and bounded rectangle if there are two vectors $a, b \in \mathbb{R}^l$ such that$a \leq b$ and $Z = [a_1, b_1] \times \ldots \times [a_l, b_l]$.

Moreover, the Theorem establishes the converges to $\lambda^\star$ from any initial distribution λ_0 by iteratively applying the adjoint Markov operator (as implied by the contraction mapping theorem for the Bellman operator).

Part II

Dynamic Programming

This part of the course develops the mathematical tools and notation used in dynamic programming problems of the type encountered in applications across macroeconomics. necessary to study how random variables affect optimization problems.

We start with an overview of non-stochastic (or deterministic) dynamic programming problems and the main mathematical results that allow us to solve them. The exposition follows almost verbatim Stokey, Lucas, and Prescott (1989, Ch. 3). The main objective here is just having a reference for the main results in dynamic programming.

Having established the basics we extend the notation and the basic results to be able to deal with stochastic problems. This follows Stokey, Lucas, and Prescott (1989, Ch. 9). The objective of this section is to mention the main results without getting into details as they are all extensions of the non-stochastic case.

Finally, we will use these tools to tackle centralized and decentralized economies in the context of the Neoclassical Growth Model. We will first layout the decentralized version of the model and define equilibrium. Having done that we can discuss how to solve the model using the centralized problem. Finally we discuss the nature of the solution in terms of Markov chains and their properties.

6. Non-Stochastic Dynamic Programming [Optional]

The topic of this section is how to state and solve (deterministic) dynamic programming problems. That is how to solve a Bellman equation of the form:

$$v(x) = \sup_{y \in \Gamma(x)} \left\{ F(x, y) + \beta v(y) \right\}$$

where the solution is given by a function v satisfying the equation. The final objective is to establish conditions for a solution to exist and characterize the properties of such a solution. To do this some mathematical background has to be set up, this is done in Section 6.1 where the contraction mapping theorem is stated and proven, and sufficient conditions for an operator to be a contraction are established.

Once the basic tools are in place the problem at hand is to express usual sequential problems (stated in terms of infinite sums) in a recursive way, the equivalence between the two representations of the problem is established by optimality principle which is presented in Section 6.2, along with it the conditions for existence of a solution and the properties it can inherit from the objective function F and the correspondence Γ are listed.

All the exposition of the theoretical aspects follows (very) closely Section 3.2 and all of chapter 4 of Stokey, Lucas, and Prescott (1989).[6] Most proofs are relegated to the book because their treatment would require more time than the one the course has.

6.1. Contraction Mapping Theorem

Three results are covered in this section that will be essential for studying dynamic programming (DP) problems. These results are the contraction mapping theorem, its corollary and the Blackwell sufficiency conditions. Before stating them recall the definition of a complete metric space and of a contraction mapping (or simply contraction) in a metric space:

Definition 6.1. A metric space is a pair (S, ρ) of a set and a metric (or distance) $\rho : S \times S \to \mathbb{R}$ such that for all $x, y, z \in S$:

(a) $\rho(x, y) \geq 0$ and $\rho(x, y) = 0 \iff x = y$.

(b) $\rho(x, y) = \rho(y, x)$.

(c) $\rho(x, z) \leq \rho(x, y) + \rho(y, z)$.

[6] There is no reason to deviate.

A metric space is furthermore complete if all Cauchy sequences in S converge to an element in S.

Definition 6.2. Let (S, ρ) be a metric space and $T : S \to S$ a function mapping S into itself. T is a contraction (with modulus β) if and only if there exists $\beta \in (0, 1)$ such that for all $x, y \in S$:

$$\rho(Tx, Ty) \leq \beta \rho(x, y)$$

The iterates of T are the mappings $\{T^n\}$ defined by $T^0 x = x$ and $T^n x = T\left(T^{n-1} x\right)$ fro $n = 1, 2, \ldots$.

The contraction mapping theorem establishes the existence and uniqueness of a fixed point in S for any contraction mapping, moreover it provides a simple algorithm to approximate the fixed point from any arbitrary point in the space. A fixed point is a point $x \in S$ such that $x = Tx$.

Theorem 6.1. ***(Contraction Mapping Theorem)*** *Let (S, ρ) be a complete metric space and $T : S \to S$ a contraction with modulus β, then:*

(a) *T has exactly one fixed point $v \in S$.*

(b) *For any $v_0 \in S$ and $n = 0, 1, \ldots$ it holds that:*

$$\rho\left(T^n v_0, v\right) \leq \beta^n \rho(v_0, v)$$

Proof. The outline of the proof is to establish that the sequence $\{v_n\} \subset S$ with $v_n = T^n v_0$ is Cauchy and the use completeness of the space to argue that its limit is the fixed point of the mapping.

Let $v_0 \in S$ and define $v_{n+1} = Tv_n$ so that $v_n = T^n v_0$. Because T is a contraction mapping:

$$\rho(v_2, v_1) = \rho(Tv_1, Tv_0) \leq \beta \rho(v_1, v_0)$$

By induction we get:

$$\rho(v_{n+1}, v_n) \leq \beta^n \rho(v_1, v_0)$$

Then for $m > n$ we get:

$$\begin{aligned}
\rho(v_m, v_n) &\leq \rho(v_m, v_{m-1}) + \rho(v_{m-1}, v_{m-2}) + \ldots + \rho(v_{n+1}, v_n) \\
&\leq \left(\beta^{m-1} + \beta^{m-2} + \ldots + \beta^n\right) \rho(v_1, v_0) \\
&= \beta^n \left(\beta^{m-n-1} + \beta^{m-n-2} + \ldots + 1\right) \rho(v_1, v_0) \\
&\leq \frac{\beta^n}{1-\beta} \rho(v_1, v_0)
\end{aligned}$$

Because $\frac{\rho(v_1,v_0)}{1-\beta}$ is fixed, and finite, and $\beta^n \to 0$ its clear that for any $\epsilon > 0$ there exists N large enough for $\rho(v_m, v_n) \leq \epsilon$ for all $m, n \geq N$. Then $\{v_n\}$ is Cauchy and there exists $v \in S$ such that $v_n \to v$ because S is complete.

Now we show that v is a fixed point of T. For all n and v_0:

$$\begin{aligned}\rho(Tv, v) &\leq \rho\left(Tv, T^n v\right) + \rho\left(T^n v, v\right) \\ &\leq \beta\rho\left(v, T^{n-1}v\right) + \rho\left(T^n v, v\right) \\ &= \beta\rho(v, v_{n-1}) + \rho(v_n, v)\end{aligned}$$

It follows that $\rho(v, v_{n-1}) \to 0$ and $\rho(v_n, v) \to 0$ because $v_n \to v$. Because this is done for arbitrary n we get $\rho(Tv, v) \leq \epsilon$ for all $\epsilon > 0$ which implies $\rho(Tv, v) = 0$. By definition this is $Tv = v$, a fixed point.

To show uniqueness suppose for a contradiction that there is $v' \neq v$ such that $Tv' = v'$, then:

$$0 \neq \rho\left(v', v\right) = \rho\left(Tv', Tv\right) \leq \beta\rho\left(v', v\right)$$

But this contradicts $\beta < 1$. Then v is the unique fixed point.

The second part of the theorem follows by induction. For $n = 0$,

$$\rho(Tv_0, v) = \rho(Tv_0, Tv) \leq \beta\rho(v_0, v)$$

and for any $n \geq 1$,

$$\rho\left(T^n v_0, v\right) = \rho\left(T^n v_0, Tv\right) \leq \beta\rho\left(T^{n-1}v_0, v\right)$$

The result follows.

□

The contraction mapping theorem is a very powerful and simple theorem, yet its results can be strengthened by further characterizing the fixed point. So far it has been established its existence in S and its uniqueness, the following corollary to the theorem allows to locate the fixed point in a given subset of S.

Corollary 6.1. *Let (S, ρ) be a complete metric space and $T : S \to S$ a contraction mapping with fixed point $v \in S$.*

(a) *If $S' \subseteq S$ is closed and $T\left(S'\right) \subseteq S'$, then $v \in S'$.*

(b) *If in addition there exists $S'' \subseteq S'$ such that $T\left(S'\right) \subseteq S''$, then $v \in S''$.*

Proof. Let $v_0 \in S'$ and $\left\{T^n v_0\right\}$ a sequence in S' so that $T^n v_0 \to v$, because S' is closed it follows that $v \in S'$. If in addition $T\left(S'\right) \subseteq S''$ then it follows that $v = Tv \in S''$.

□

Finally a set of sufficiency conditions are established for a mapping on the space of bounded functions to be a contraction. In most economic applications these conditions are trivial to check.

Theorem 6.2. ***(Blackwell conditions)*** *Let $X \subseteq \mathbb{R}^l$ and $B(X)$ be the the space of bounded functions on X ($f : X \to \mathbb{R}$) with the sup-norm. Let $T : B(X) \to B(X)$, T is a contraction (with modulus β) if it satisfies the following two conditions:*

(a) *(monotonicity) Let $f, g \in B(X)$ and $f(x) \leq g(x)$ for all $x \in X$. Then $Tf(x) \leq Tg(x)$ for all $x \in X$.*

(b) *(discounting) There exists $\beta \in (0,1)$ such that $T(f+a)(x) \leq Tf(x) + \beta a$ for all $f \in B(X)$, $x \in X$ and $a \geq 0$.*

Proof. If $f(x) \leq g(x)$ for all x we say that $f \leq g$.

Let $f, g \in B(X)$, by definition of the sup-norm $f(x) - g(x) \leq \|f - g\|$ for all $x \in X$, then $f(x) \leq g(x) + \|f - g\|$, using the notation defined at the beginning of the proof this is $f \leq g + \|f - g\|$, where $\|f - g\| > 0$ is a scalar. Then by hypothesis we have:

$$Tf \leq T(g + \|f - g\|) \leq Tg + \beta \|f - g\| \longrightarrow Tf - Tg \leq \beta \|f - g\|$$

But it also holds that $g(x) - f(x) \leq \|f - g\|$ which implies $Tg - Tf \leq \beta \|f - g\|$. Joining we have, for all $x \in X$:

$$|Tf(x) - Tg(x)| \leq \beta \|f - g\|$$

Taking sup we get:

$$\|Tf - Tg\| \leq \beta \|f - g\|$$

which establishes that T is a contraction.

□

Extended Blackwell conditions. I also present a modified version of Blackwell's sufficiency conditions for vector valued functions. I first define the relevant set of functions.

Proposition 6.1. *Let $X \subset \mathbb{R}^n$ and $B(X) = \left\{ f | f : X \to \mathbb{R} \wedge \exists_{M_f} \forall_{x \in X} |f(x)| \leq M_f \right\}$ the set of bounded functions defined on the set X. The space $S = B(X) \times B(X)$ equipped with the norm $\|f\| = \max\left\{ \|f_1\|_\infty, \|f_2\|_\infty \right\} = \max\left\{ \sup_{x \in X} |f_1(x)|, \sup_{x \in X} |f_2(x)| \right\}$ is a normed vector space. It is also a metric space with the metric $\rho(f, g) = \|f - g\|$.*

Proof. The proof proceeds by showing that $\|\cdot\|$ is a norm.

(a) Clearly $\|f\| \geq 0$ and if $f(x) = 0$ for all $x \in X$ then $\|f\| = 0$. Finally:

$$\begin{aligned} \|f\| &= 0 \\ \max\left\{\sup_{x\in X}|f_1(x)|, \sup_{x\in X}|f_2(x)|\right\} &= 0 \end{aligned}$$

which happens if and only if $\sup_{x\in X}|f_1(x)| = 0$ and $\sup_{x\in X}|f_2(x)| = 0$. Again, this happens if and only if $f_1(x) = f_2(x) = 0$ for all $x \in X$. That is, if $f(x) = 0$ for all $x \in X$.

(b) $\|\alpha f\| = \max\left\{\sup_{x\in X}|\alpha f_1(x)|, \sup_{x\in X}|\alpha f_2(x)|\right\} = |\alpha| \max\left\{\sup_{x\in X}|f_1(x)|, \sup_{x\in X}|f_2(x)|\right\} = |\alpha|\,\|f\|$

(c) Triangle Inequality:

$$\begin{aligned} \|f+g\| &= \max\left\{\sup_{x\in X}|f_1(x)-g_1(x)|, \sup_{x\in X}|f_2(x)-g_2(x)|\right\} \\ &\leq \max\left\{\left(\sup_{x\in X}|f_1(x)| + \sup_{x\in X}|g_1(x)|\right), \left(\sup_{x\in X}|f_2(x)| + \sup_{x\in X}|g_2(x)|\right)\right\} \\ &\leq \max\left\{\sup_{x\in X}|f_1(x)|, \sup_{x\in X}|f_2(x)|\right\} + \max\left\{\sup_{x\in X}|g_1(x)|, \sup_{x\in X}|g_2(x)|\right\} \\ &= \|f\| + \|g\| \end{aligned}$$

The first inequality follows from properties of the absolute value and the second one from the inequality:

$$\sup_{x\in X}|f_i(x)| + \sup_{x\in X}|g_i(x)| \leq \max\left\{\sup_{x\in X}|f_1(x)| + \sup_{x\in X}|g_1(x)|, \sup_{x\in X}|f_2(x)| + \sup_{x\in X}|g_2(x)|\right\}$$

(d) Under the above three conditions $\|f\|$ is a norm.

(e) Clearly the sum and scalar product of bounded functions is bounded.

□

Proposition 6.2. *Consider* (S, ρ) *with* $S = B(X) \times B(X)$ *and* $\rho(f,g) = \|f-g\|$. (S, ρ) *is a complete space.*

Proof. The proof starts by showing that a Cauchy sequence in S is formed by Cauchy sequences in $B(X)$. Then the completeness of $B(X)$ is used to establish the result.

(a) Let $\{f_n\} \subset S$ be a Cauchy sequence and $\epsilon > 0$. There exists N such that $\forall_{n,m>N} \|f_n - f_m\| < \epsilon$ which is:

$$\max\left\{\sup_{x\in X}|f_{1n}(x) - f_{1m}(x)|, \sup_{x\in X}|f_{2n}(x) - f_{2m}(x)|\right\} < \epsilon$$

$$\sup_{x\in X}\left|f_{1n}(x)-f_{1m}(x)\right|<\epsilon \qquad \wedge \qquad \sup_{x\in X}\left|f_{2n}(x)-f_{2m}(x)\right|<\epsilon$$

This implies that the sequences $\{f_{1n}\}\subset B(X)$ and $\{f_{2n}\}\subset B(X)$ are Cauchy with respect to the sup-norm $\|f\|_\infty=\sup_{x\in X}|f(x)|$.

(b) Because the space $(B(X),\|f\|_\infty)$ is complete, the above implies that the sequences $\{f_{1n}\}$ and $\{f_{2n}\}$ are convergent in the sup norm. $\exists_{f_1,f_2\in B(X)} f_{1n}\to f_1\wedge f_{2n}\to f_2$. Denote $f:S\to\mathbb{R}^2$ as $f(x)=\left[f_1(x)\ \ f_2(x)\right]'$.

(c) Let $\epsilon>0$. By convergence of $\{f_{1n}\}$ and $\{f_{2n}\}$ there exist numbers N_1 and N_2 such that:

$$\forall_{n\geq N_1}\left\|f_{1n}-f_1\right\|_\infty<\epsilon \qquad \wedge \qquad \forall_{n\geq N_2}\left\|f_{2n}-f_2\right\|_\infty<\epsilon$$

Then for $N=\max\{N_1,N_2\}$ it holds that:

$$\forall_{n\geq N}\left\|f_{1n}-f_1\right\|_\infty<\epsilon \qquad \wedge \qquad \left\|f_{2n}-f_2\right\|_\infty<\epsilon$$

which is:

$$\forall_{n\geq N}\sup_{x\in X}\left|f_{1n}(x)-f_1(x)\right|<\epsilon \qquad \wedge \qquad \sup_{x\in X}\left|f_{2n}(x)-f_2(x)\right|<\epsilon$$

implying then:

$$\forall_{n\geq N}\max\left\{\sup_{x\in X}\left|f_{1n}(x)-f_1(x)\right|,\sup_{x\in X}\left|f_{2n}(x)-f_2(x)\right|\right\}<\epsilon$$

which is:

$$\forall_{n\geq N}\left\|f_n-f\right\|<\epsilon$$

(d) The above proves that a Cauchy sequence converges on S over the given norm.

□

Theorem 6.3. ***(Extended Blackwell)*** *Consider (S,ρ) with $S=B(X)\times B(X)$ and $\rho(f,g)=\|f-g\|$. Let $T:S\to S$ be an operator satisfying*

(a) *(Monotonicity) $f,g\in S$ and $f(x)\leq g(x)$, for all $x\in X$, implies $Tf(x)\leq Tg(x)$, for all $x\in X$, (where $f(x)\leq g(x)$ is taken in the vector sense, i.e. $f_1(x)\leq g_1(x)$ and $f_2(x)\leq g_2(x)$).*

(b) *(Discounting) there exists some $\beta\in(0,1)$ such that $T(f+A)\leq Tf(x)+\beta A$ for $f\in S$, $A=[a\ a]'\in\mathbb{R}^2_+$ and $x\in X$.*

Then T is a contraction in S with modulus β.

Proof. The proof follows closely that of Blackwell's conditions

(a) Let $f, g \in S$, and define $A = \begin{bmatrix} \|f-g\| & \|f-g\| \end{bmatrix}'$, it holds that:

$$\begin{aligned} f_1(x) - g_1(x) &\leq |f_1(x) - g_1(x)| \leq \sup_{x \in X} |f_1(x) - g_1(x)| \\ &\leq \max\left\{ \sup_{x \in X} |f_1(x) - g_1(x)|, \sup_{x \in X} |f_2(x) - g_2(x)| \right\} = \|f-g\|_\infty \end{aligned}$$

By a similar argument $f_2(x) - g_2(x) \leq \|f-g\|_\infty$ then it holds that: $f(x) \leq g(x) + A$ for all $x \in X$

(b) By monotonicity and discounting:

$$Tf(x) \leq T(g+a)(x) \leq Tg(x) + \beta A$$

which holds for all $x \in X$.

(c) The same argument applies to show that $g(x) \leq f(x) + A$ and $Tg(x) \leq Tf(x) + \beta A$ for all $x \in X$.

(d) Joining:

$$Tf_i(x) - Tg_i(x) \leq \beta \|f-g\| \qquad \wedge \qquad Tg_i(x) - Tf_i(x) \leq \beta \|f-g\|$$

which implies:

$$|Tf_1(x) - Tg_1(x)| \leq \beta \|f-g\| \qquad \wedge \qquad |Tf_2(x) - Tg_2(x)| \leq \beta \|f-g\|$$

and then:

$$\sup_{x \in X} |Tf_1(x) - Tg_1(x)| \leq \beta \|f-g\| \qquad \wedge \qquad \sup_{x \in X} |Tf_2(x) - Tg_2(x)| \leq \beta \|f-g\|$$

(e) Finally:

$$\begin{aligned} \|Tf - Tg\|_\infty &= \max\left\{ \sup_{x \in X} |Tf_1(x) - Tg_1(x)|, \sup_{x \in X} |Tf_2(x) - Tg_2(x)| \right\} \\ &\leq \max\left\{ \beta \|f-g\|, \beta \|f-g\| \right\} = \beta \|f-g\| \end{aligned}$$

This is the definition of T being a contraction with modulus β.

□

6.2. The Bellman Equation

We start with the infinite horizon consumption savings model. This is the workhorse model of modern macroeconomics and is known as the neoclassical growth model. There are two (related) ways of setting up the problem. One resembles the finite horizon problem already discussed, it is called sequence problem, the other form is to cast the problem as the solution to a functional equation, this dynamic programming approach has several advantages that will be presented in the next section.

As before, consider a discrete time, consumption-savings problem where the agent can either consume or save (invest) in capital that will be productive in the following period. The agent derives utility from consumption according to utility function u and discounts the future at a constant rate $\beta < 1$. Production only uses capital and the technology is described by a function f.

The problem of an agent endowed with k_0 units of capital is:

$$v(k_0) = \max_{\{c_t, k_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(c_t) \qquad \text{s.t. } c_t + k_{t+1} \leq f(k_t) \quad c_t, k_t \geq 0 \quad k_0 \text{ given}$$

Provided that u is strictly increasing, a sustained assumption, we can eliminate consumption as before to get:

$$v(k_0) = \max_{\{k_{t+1}\}} \sum_{t=0}^{\infty} \beta^t u(f(k_t) - k_{t+1}) \qquad \text{s.t. } 0 \leq k_{t+1} \leq f(k_t) \quad k_0 \text{ given}$$

In the sequence problem, much like in the finite horizon problem before, the objective is to look for an infinite sequence that solves the problem and attains the maximum. This can prove to be too difficult in practice.

The dynamic programming problem takes a different approach. Instead of trying to solve the problem for all periods simultaneously the objective is to solve the problem one period at a time. That is, given the capital stock at the beginning of the period take an optimal investment decision for the next period. The problem is that, in order to make the decision, its necessary to know the extra value for the agent of the capital to be saved, we need a function that represents preferences over next period's capital.

The DP starts by assuming that we already know such a function. It is called a value function and is defined as v above. The value function is the maximum value given to the agent if she starts in a given period with initial capital k. Knowing v it is possible to cast the following problem:

$$\max_{0 \leq k_1 \leq f(k_0)} \left\{ u(f(k_0) - k_1) + \beta v(k_1) \right\}$$

If we knew v the problem above could be solved. The solution to the problem is a policy function $g : \mathbb{R}_+ \to \mathbb{R}_+$ that gives the optimal capital next period given a capital level today. That is $k_1 = g(k_0)$.

It should be clear now that if $v(k_1)$ gives the maximum value starting in period 1 and the problem above maximizes that value and the value in period 0 (given by $u(f(k_0) - k_1)$) then the value of the whole problem is given by the maximum above. But that is the definition

of v, then:

$$v(k_0) = \max_{0 \leq k_1 \leq f(k_0)} \left\{ u(f(k_0) - k_1) + \beta v(k_1) \right\}$$

This is a functional equation, where f and u are known functions, k_1 is a variable of choice and k_0 is given. Then this is an equation in the function v, the solution to this equation is the value function needed to solve the problem (to find the policy function).

In general solving functional equations is not easy, but this type of functional equation can be reinterpreted to both establish the existence of a solution and to obtain a method to find it.

Let u and f be bounded and continuous functions and define an operator $T : C(X) \to C(X)$ as:

$$Tv(k) = \max_{0 \leq k' \leq f(k)} \left\{ u\left(f(k) - k'\right) + \beta v\left(k'\right) \right\}$$

boundedness of Tv is immediate for the sum of bounded is also bounded. Continuity is a consequence of the ToM, the objective function is continuous and the correspondence $\Gamma(k) = \left\{ k' | 0 \leq k' \leq f(k) \right\}$ is continuous and compact valued because f is continuous and bounded.

The solution to the functional equation, v, is then a fixed point of the mapping T. It is left to verify that T is a contraction to establish the existence and uniqueness of the solution to the neoclassical growth model. It turns out that Blackwell's sufficient conditions are immediate:

(a) (monotonicity) Let $v, w \in C(X)$ and $v(k) \leq w(k)$ for all k. Then:

$$Tv(k) = \max_{0 \leq k' \leq f(k)} \left\{ u\left(f(k) - k'\right) + \beta v\left(k'\right) \right\} \leq \max_{0 \leq k' \leq f(k)} \left\{ u\left(f(k) - k'\right) + \beta w\left(k'\right) \right\} = Tw(k)$$

(b) (discounting) Let $v \in C(X)$ and $a > 0$. Then:

$$T(v + a)(k) = \max_{0 \leq k' \leq f(k)} \left\{ u\left(f(k) - k'\right) + \beta \left(v\left(k'\right) + a \right) \right\} = \max_{0 \leq k' \leq f(k)} \left\{ u\left(f(k) - k'\right) + \beta v\left(k'\right) \right\} + \beta a$$

In particular:

$$T(v + a)(k) \leq Tv(k) + \beta a$$

It is possible to further characterize v and the policy function g, for that extra results are needed.

6.3. A general framework and the principle of optimality

The problem to be studied in terms of infinite sequences is of the form:

$$v^{\star}(x_0) = \sup_{\{x_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t F(x_t, x_{t+1}) \qquad \text{s.t. } x_{t+1} \in \Gamma(x_t) \tag{6.1}$$

Corresponding to this problem is the following functional equation:

$$v(x) = \sup_{y \in \Gamma(x)} \left\{ F(x, y) + \beta v(y) \right\} \tag{6.2}$$

Above, X is the set of possible values for x, and it is not necessarily an euclidean space, $\Gamma : X \rightrightarrows X$ is a correspondence that assigns feasible values of the choice variable and $F : \text{Gr}(\Gamma) \to \mathbb{R}$ is a return or payoff function. $\beta > 0$ is a discount factor.

Some conditions have to be met for both problems to give the same solution, in the sense that $v(x) = v^{\star}(x)$ and that the optimal choice of one problem is the the same as the choice for the other. This equivalence between both problems is called the *principle of optimality*. After the validity of the principle has been established the properties of the solution to FE can be studied.

The conditions for the principle of optimality are stated below and the two propositions that constitute the principle are shown without proof.

It will be convenient to define the set of all possible feasible sequences for x, given an starting point x_0.

Definition 6.3. The set of all possible feasible sequences starting at $x_0 \in X$ is:

$$\Pi(x_0) = \left\{ \{x_t\}_{t=0}^{\infty} \,|\, x_{t+1} \in \Gamma(x_t) \quad \wedge \quad x_0 \text{ given} \right\}$$

and $\underline{x} = (x_0, x_1, x_2, \ldots)$ is an element.

Assumption A.1: . Γ is a nonempty valued correspondence.

Assumption A.2: . For all $x_0 \in X$ and $\underline{x} \in \Pi(x_0)$ the following limit exists (although it might be infinite):

$$\lim_{n \to \infty} \sum_{t=0}^{n} \beta^t F(x_t, x_{t+1})$$

Remark. Assumption A.2 holds if F is bounded and $\beta \in (0, 1)$.

Under assumptions A.1 and A.2 $\Pi(x_0)$ is nonempty valued and problem (6.1) is well posed, moreover they are enough to guarantee that the function $v^\star$ satisfies equation (6.2).

Proposition 6.3. *Let X, Γ, F and β satisfy assumption A.1 and A.2, then $v^\star$ is a solution to the FE (6.2):*

$$v^\star(x) = \sup_{y \in \Gamma(x)} \left\{ F(x, y) + \beta v^\star(y) \right\}$$

For $v^\star$ to be the only solution to the FE an extra condition is needed.

Proposition 6.4. *Let X, Γ, F and β satisfy assumption A.1 and A.2, if v is a solution to the FE (6.2) and for all $x_0 \in X$ and $\underline{x} \in \Pi(x_0)$ it holds that:*

$$\lim_{n \to \infty} \beta^n v(x_n) = 0$$

then $v = v^\star$.

The previous two propositions establish equivalence between the value of the two problems. It can also be shown that the optimizer of the SP problem also solves the FE in the following sense:

Proposition 6.5. *Let X, Γ, F and β satisfy assumption A.1 and A.2. Let $\underline{x}^\star \in \Pi(x_0)$ be a feasible plan that attains the supremum in (6.1), then:*

$$v^\star\left(x_t^\star\right) = F\left(x_t^\star, x_{t+1}^\star\right) + \beta v^\star\left(x_{t+1}^\star\right) \tag{6.3}$$

Again, under an extra boundedness condition a plan that solves the problem in (6.2) also solves the problem in the SP.

Proposition 6.6. *Let X, Γ, F and β satisfy assumption A.1 and A.2. Let $\underline{x}^\star \in \Pi(x_0)$ be a feasible plan that satisfies equation (6.3) and for which $\limsup \beta^t v^\star\left(x_t^\star\right) \leq 0$, then $\underline{x}^\star$ attains the supremum in (6.1) for initial state x_0.*

Now we can define the optimal policy correspondence as:

$$G^\star(x) = \left\{ y \in \Gamma(x) \,|\, v^\star(x) = F(x, y) + \beta v^\star(y) \right\}$$

We say that a plan $\underline{x}$ is generated by G if it satisfies $x_{t+1} \in G(x_t)$. The previous two propositions imply that any optimal plan of the sequence problem is generated by $G^\star$ and that if a plan is generated by $G^\star$ and satisfies the additional boundedness condition then it is also optimal.

Now we can concentrate in studying the properties of the DP in (6.2).

6.4. Bounded problems

Now we concentrate in establishing properties of the solution to the following problem:

$$v(x) = \max_{y \in \Gamma(x)} \left\{ F(x, y) + \beta v(y) \right\} \tag{6.4}$$

$$G(x) = \left\{ y \in \Gamma(x) \,|\, v(x) = F(x, y) + \beta v(y) \right\}$$

where v is the value function and G the policy correspondence.

Assumptions A.1 and A.2 have to be met for the implications of this sections to be valid on the original sequence problem. Additional assumptions are also imposed that ensure that the previous ones are met.

Assumption A.3: . X is a a convex subset of $\mathbb{R}^l$ and Γ is a nonempty, compact valued and continuous correspondence.

Assumption A.4: . The function $F : \mathrm{Gr}(\Gamma) \to \mathbb{R}$ is bounded and continuous and $\beta \in (0, 1)$.

Because F is bounded and continuous it is natural to think that the solution to equation (6.4) lies in the set $C(X)$. What follows it to establish the existence of a solution by means of the contraction mapping theorem.

Define a mapping $T : C(X) \to C(X)$ as:

$$Tf(x) = \max_{y \in \Gamma(x)} \left\{ F(x, y) + \beta f(y) \right\} \tag{6.5}$$

The solution to (6.4) is then a $v \in C(X)$ such that $v = Tv$. The following proposition establishes that T is a contraction from $C(X)$ into itself and also some properties of the policy correspondence G.

Proposition 6.7. *Let X, Γ, F and β satisfy assumption A.3 and A.4, and consider $C(X)$ the space of continuous bounded function on X along with the sup norm. Then:*

(a) *T defined in (6.5) maps $C(X)$ into itself.*

(b) *T defined in (6.5) has a unique fixed point $v \in C(X)$, and for all $v_0 \in C(X)$*

$$\left\| T^n v_0 - v \right\| \leq \beta^n \left\| v_0 - v \right\|$$

(c) *Given v the optimal policy correspondence $G(x) = \left\{ y \in \Gamma(x) \,|\, v(x) = F(x, y) + \beta v(y) \right\}$ is nonempty, compact valued and u.h.c.*

Proof. Each part is established separately.

(a) Under A.3 and A.4 and given f continuous and bounded the function $F(x, y) + \beta f(y)$ is continuous in (x, y) and Γ satisfies all assumptions of the ToM, thus establishing that Tf is continuous.

Because F and f are bounded then Tf is bounded as well. There exists $M \geq 0$ such that $-M \leq F(x, y) + \beta f(y) \leq M$ for all (x, y), then for all x we have: $-M \leq \max_{y \in \Gamma(x)} \{F(x, y) + \beta f(y)\} \leq M$ which establishes boundedness of Tf.

Then $Tf \in C(X)$ for any $f \in C(X)$.

(b) Blackwell conditions are met:

(i) (monotonicity) Let $f, g \in C(X)$ and $f(x) \leq g(x)$ for all x. Then:

$$Tf(x) = \max_{y \in \Gamma(x)} \{F(x, y) + \beta f(y)\} \leq \max_{y \in \Gamma(x)} \{F(x, y) + \beta g(y)\} = Tg(x)$$

(ii) (discounting) Let $f \in C(X)$ and $a > 0$. Then:

$$T(f + a)(x) = \max_{y \in \Gamma(x)} \{F(x, y) + \beta (f(y) + a)\} = \max_{y \in \Gamma(x)} \{F(x, y) + \beta f(y)\} + \beta a$$

In particular:

$$T(f + a)(x) \leq Tf(x) + \beta a$$

Then T is a contraction. By the contraction mapping theorem the result follows.

(c) The properties of G follow from the ToM which applies as shown before.

□

Additional assumption will help to characterize v and G better. The corollary of the contraction mapping theorem is the tool to be used now. First monotonicity can be inherited by the solution.

Assumption A.5:. For all y $F(\cdot, y)$ is strictly increasing in its first l arguments.

Assumption A.6:. Γ is monotone in the sense that if $x \leq x'$ the $\Gamma(x) \subseteq \Gamma\left(x'\right)$.

Proposition 6.8. *Let X, Γ, F and β satisfy assumption A.3 to A.6, and let v be the unique solution to (6.4), then v is strictly increasing.*

Proof. Let $C'(X) \subseteq C(X)$ be the set of bounded, continuous and non-decreasing functions and $C''(X) \subseteq C'(X)$ the set of strictly increasing functions. Clearly $C'(X)$ is closed. By the corollary of the contraction mapping theorem it suffices to show that $T\left(C'(X)\right) \subseteq C''(X)$.

Let $f \in C'(X)$ and consider $x < x'$. We want to show that Tf is strictly increasing. This follows with A.5 and A.6:

$$Tf(x) = \max_{y \in \Gamma(x)} F(x, y) + \beta f(y) \leq \max_{y \in \Gamma\left(x'\right)} F(x, y) + \beta f(y) < \max_{y \in \Gamma\left(x'\right)} F\left(x', y\right) + \beta f(y) = Tf\left(x'\right)$$

where the first inequality follows from $\Gamma(x) \subseteq \Gamma\left(x'\right)$, a larger choice set implies a higher than or equal maximum, the second inequality follows from F being strictly increasing.

□

It is also possible to induce convexity as follows:

Assumption A.7: . F is strictly concave in both arguments.

Assumption A.8:. Γ has a convex graph.

Proposition 6.9. *Let X, Γ, F and β satisfy assumption A.3, A.4, A.7 and A.8, and let v be the unique solution to (6.4), then v is strictly concave and G is a continuous single valued function.*

Proof. Let $C'(X)$ be the set of concave, bounded and continuous functions and $C''(X)$ the set of strictly concave, bounded and continuous functions. The set $C'(X) \subseteq C(X)$ is closed and that $C''(X) \subseteq C'(X)$. To show that v is strictly concave we use the corollary of the contraction mapping theorem.

We want to show that for all $f \in C'(X)$ it follows that $Tf \in C''(X)$ where

$$Tf(x) = Tf(x) = \max_{y \in \Gamma(x)} F(x, y) + \beta f(y)$$

So let f be weakly concave on x, bounded and continuous. let $x_1, x_2 \in X$ and $\lambda \in (0, 1)$ and define $x_\lambda = \lambda x_1 + (1 - \lambda) x_2$. Let $y_i \in G\left(x_i\right) \subseteq \Gamma\left(x_i\right)$ and, by A.8, $y_\lambda = \lambda y_1 + (1 - \lambda)\, y_2 \in \Gamma(x_\lambda)$:

$$\begin{aligned}
Tf(x_\lambda) &= \max_{y \in \Gamma(x_\lambda)} F(x_\lambda, y) + \beta f(y) \\
&\geq F\left(x_\lambda, y_\lambda\right) + \beta f\left(y_\lambda\right) \\
&\geq F\left(x_\lambda, y_\lambda\right) + \lambda \beta f\left(y_1\right) + (1 - \lambda)\, \beta f\left(y_2\right) \\
&> \lambda F\left(x_1, y_1\right) + (1 - \lambda)\, F\left(x_2, y_2\right) + \lambda \beta f\left(y_1\right) + (1 - \lambda)\, \beta f\left(y_2\right) \\
&= \lambda T f(x_1) + (1 - \lambda)\, T f(x_2)
\end{aligned}$$

where the first inequality follows from $y_\lambda$being feasible at x_λ, the second one from f being concave and the third one from A.7. The final equality is obtained rearranging and recalling the optimality of y_1 and y_2 under x_1 and x_2 respectively. Joining results we get

$$Tf(x_\lambda) > \lambda T f(x_1) + (1 - \lambda)\, T f(x_2)$$

Then the "image" of any concave function is a strictly concave function. This proves that $Tf \in C''(X)$. Then by the corollary of the contraction mapping theorem the unique fix point of T belongs to C''. That is, v is strictly concave.

Finally, consider the policy function $G(x) = \underset{y \in \Gamma(x)}{\operatorname{argmax}} F(x, y) + v(y)$. Because F and v are strictly concave the single valuedness and continuity of G follow as an immediate consequence of the ToM under convexity part (ii).

□

Finally there are conditions for v to be differentiable, allowing the use of first order conditions.

Assumption A.9: . *F* is continuously differentiable on the interior of its domain, Gr (A).

Proposition 6.10. *Let X, Γ, F and β satisfy assumption A.3, A.4, and A.7 to A.9, and let v be the unique solution to (6.4). If $x_0 \in IntX$ and $g(x_0) \in Int\Gamma(x_0)$ then v is continuously differentiable at x_0 with derivatives given by:*

$$v_i(x_0) = F_i(x_0, g(x_0))$$

Proof. Stokey, Lucas, and Prescott (1989, sec. 4.2, pp. 85).

□

7. Stochastic Dynamic Programming

We now want to establish the basics of dynamic programming problems where (some of) the states are subject to stochastic variation. We want to allow the variation in the stochastic variables to be serially correlated and for that we use the concepts developed for Markov processes in Section 5. The results below are a summary of those in Chapter 9 of Stokey, Lucas, and Prescott (1989).

The basic problem takes one of two forms:

$$v(k,z) = \max_{k' \in [0, zf(k)]} \left\{ U\left(zf(k) - k'\right) + \beta \int_z v\left(k', z'\right) Q\left(z, dz'\right) \right\} \tag{7.1}$$

$$v(k,z) = \max_{y \in [0, f(k)]} \left\{ U\left(f(k) - y\right) + \beta \int_z v\left(y - z', z'\right) Q\left(z, dz'\right) \right\} \tag{7.2}$$

The key mathematical feature in the two models is the expectation over future values of the state z taking as given the state's current value. The conditional distribution of z' is given by the transition function Q.

The key economic difference between the two problems is the degree of control over the future value of the states. In the first problem the future value of state k is perfectly controlled and there is only uncertainty over the future value of z. In the second problem the value of the first state is only known until the following period. Only the choice y is known by the decision maker, but the value of $k' = y - z'$ depends on z'.

The second problem is of course more general and it also allows for more flexible numerical methods for the solution of the functional equation. See Phelan and Eslami (2022).

The general form of the stochastic dynamic programming problem is cumbersome because of the required notation to take into account the stochastic process induced by the choices of the decision maker. However, the end result is basically the same as in the non-stochastic case of Section 6. Most importantly, the principle of optimality and the contraction mapping theorem still apply.

In what follows we cover the basic definitions that will be useful to establish what constitutes a solution to the stochastic dynamic programming problem. We will later use these to construct a competitive equilibrium in decentralized economies. We will do this for the first type of problem shown above. Even though it is less general, the basic concepts are clearer in it and it maps better to the neoclassical growth model, leading example to come.

We start by defining measurable spaces for the endogenous state $(X, \mathcal{X})$ and the

exogenous (stochastic) state $(Z, \mathcal{Z})$, as well as their product that forms the state space of the problem $(S, \mathcal{S}) = (X \times Z, \mathcal{X} \times \mathcal{Z})$. We also impose that Q is a stationary transition function, particularly that it satisfies the Feller property (definition 5.3) preserving continuity and boundedness.

The decision problem consists in choosing the future value of the endogenous state, a choice y from a feasible set characterized by the correspondence $\Gamma : X \times Z \to Z$. The graph of Γ is $A = \{(x, y, z) \,|\, y \in \Gamma(x, z)\}$. The payoff function is $F : A \to \mathbb{R}$ and the decision maker discounts future payoffs at a rate $\beta \geq 0$.

7.1. The Sequential Problem

The sequential problem is where most of the notational complications arise. The decision maker takes as given some initial state (x_0, z_0) and then makes *contingency plans* that specify actions in future periods depending on the realization of shocks up until then (the information available). Importantly, the plans depend only on the history of shocks (the exogenous state) as the endogenous state, x, is chosen every period, except for the first. The objective is to maximize the present discounted expected payoff. The expectation is over the possible sample paths for the realization of the shocks. This requires dusting off the notation on stochastic processes developed after definition 5.7.

Definition 7.1. (Contingency Feasible Plan) Let $\left(Z^t, \mathcal{Z}^t\right)$ be a measurable space over the partial history of shocks in periods 1 through t. A feasible plan is a value $\pi_0 \in \Gamma(x_0, z_0)$ and a sequence of measurable functions $\pi_t : Z^t \to \Gamma\left(\pi_{t-1}\left(z^{t-1}\right), z_t\right)$. Denote by $\Pi(s_0)$ the set of plans feasible given $s_0 = (x_0, z_0)$.

The measurability conditions over the (choice) functions π_t implies that we can compute probabilities over the outcomes (decisions) given the probabilities on paths implied by Q. Recall that Q is a function over $(Z, \mathcal{Z})$ so that, given s_0, we can define the probability measure $\mu^t(z_0, \cdot) : \mathcal{Z}^t \to [0, 1]$ as in definition 5.5. This function establishes probabilities over sequences of shocks z^t and allows to compute expectations over payoffs $F(\cdot)$ provided that F is measurable with respect to the appropriate σ-algebra on A (defined as $\mathcal{A} = \{C \in \mathcal{X} \times \mathcal{X} \times \mathcal{Z} | C \in A\}$) and that the function F is integrable (either by being always positive or negative, or by being integrable with respect to μ^t given any feasible plan $\pi \in \Pi(s_0)$). These conditions allow us to define the (period-t) discounted payoffs

$$u_0(\pi, s_0) = F(x_0, \pi_0, z_0);$$
$$u_n(\pi, s_0) = F(x_0, \pi_0, z_0) + \sum_{t=1}^{n} \beta^t \int_{Z^t} F\left(\pi_{t-1}\left(z^{-1}\right), \pi_t\left(z^t\right), z_t\right) \mu^t\left(z_0, dz^t\right);$$

and its limit

$$u(\pi, s_0) = \lim_{n\to\infty} u_n(\pi, s_0).$$

This finally allow us to define the sequential problem of the decision maker:

$$v^\star(s) = \max_{\pi\in\Pi(s)} u(\pi, s) \tag{7.3}$$

Just posing this problem is a mouthful. Solving it is even worse. First, just evaluating the probabilities along the sample path becomes increasingly taxing. For instance, even if the exogenous state could take only two values, there are over a thousand possible histories after 10 periods (and the number gets prohibitively large for 100 periods). Each of these histories has its own contingent plan, that is evaluated depending on its probability.

When the decision maker does not have perfect control over the endogenous state variable as in the second problem in (7.2) we must also define a law of motion for the endogenous state variable

$$\phi : X \times X \times Z \to X$$

so that $x' = \phi\left(x, y, z'\right)$ is the next period's endogenous state. This law of motion interacts with the contingent choices of the control, y, to generate a contingent plan, that is now

Definition 7.2. (Contingency Feasible Plan') Let $\left(Z^t, \mathcal{Z}^t\right)$ be a measurable space over the partial history of shocks in periods 1 through t. A feasible plan is a value $\pi_0 \in \Gamma(x_0, z_0)$ and a sequence of measurable functions $\pi_t : Z^t \to \Gamma\left(x_t^\pi\left(z^t\right), z_t\right)$, where $x_t^\pi\left(z^t\right) \equiv \phi\left(x_{t-1}^\pi\left(z^{t-1}\right), \pi_{t-1}\left(z^{t-1}\right), z_t\right)$ is defined as the (contingent) path of the endogenous variable given the plan π. Denote by $\Pi(s_0)$ the set of plans feasible given $s_0 = (x_0, z_0)$.

The implication of the changes to the problem is that we now have two stochastic processes. One is the exogenous state variable z, that we treat just as before. In the first type of problem this process is enough to describe the problem as it completely describes the process of choices, y, and hence of the endogenous state $x' = y$. In this more general case we have to interface the process for z through the law of motion ϕ to obtain the process for x given a plan π.

7.2. The Recursive Problem

The alternative to the sequential problem is to exploit the Markov nature of the exogenous (stochastic) state. Because z_t follows a Markov process we do not actually have to carry around the whole history of shocks and instead we can subsume it in its current realization

z_t. However, this is not enough to determine the choice of the decision maker. In the sequential problem, the sequence of the exogenous state z^t, along with the initial value of the endogenous state x_0, contained all the information needed to know the current state given a contingent plan. If we drop the history of the exogenous state in favor of just its current realization, we have to introduce the current value of x that contains the (relevant) information about the previous shocks.

The problem is now

$$v(x,z) = \max_{y \in \Gamma(x,z)} \left\{ F(x,y,z) + \beta \int v\left(y, z'\right) Q\left(z, dz'\right) \right\}, \tag{7.4}$$

with its associated policy function (or in general a correspondence)

$$G(x,z) = \left\{ y \in \Gamma(x,z) \,|\, (x,z) = F(x,y,z) + \beta \int v\left(y, z'\right) Q\left(z, dz'\right) \right\}. \tag{7.5}$$

The policy function G is the object of interest for most problems as it describes the actions of the decision maker and, as we will discuss below, it induces a distribution over the outcomes of the model at the core of representative and heterogeneous agent models.

In the second type of problem in (7.2) the recursive formulation introduces the law of motion ϕ:

$$v(x,z) = \max_{y \in \Gamma(x,z)} \left\{ F(x,y,z) + \beta \int v\left(\phi\left(x,y,z'\right), z'\right) Q\left(z, dz'\right) \right\}. \tag{7.6}$$

7.3. The Principle of Optimality

The first half of the principle of optimality comes easily. We can construct the optimal contingency plans π_t out of G. The general construction is simple but required worrying about measurability when G is a correspondence. When G is a function we can simply construct the contingent plans recursively as

$$\pi_0 = G(s_0), \ldots, \pi_t\left(z^t\right) = G\left(\pi_{t-1}\left(z^t\right), z_t\right), \ldots$$

This construction of π is immediately feasible and measurable. It is left to show that a contingent plan constructed this way satisfies the optimal value of the sequential problem, $v^\star$. The following theorem gives conditions for this, the proof is in Theorem 9.2 of Stokey, Lucas, and Prescott (1989).

Theorem 7.1. *Let $(X, \mathcal{X})$, $(Z, \mathcal{Z})$, Q, F, Γ, and β be given. Γ is non-empty valued and allows for a*

measurable selection. F is $\mathcal{A}$-measurable and integrable (see Assumption 9.2 in Stokey, Lucas, and Prescott 1989). Let $v^\star$ be as in (7.3) and v as in (7.4) such that

$$\lim_{t\to\infty} \int_{Z^t} \beta^t v\left(\pi_{t-1}\left(z^{t-1}\right), z_t\right) \mu^t\left(z_0, dz^t\right) = 0$$

for all feasible plans $\pi \in \Pi(s_0)$ and all initial states $s_0 \in X \times Z$. Finally, let G be defined as in (7.5) and suppose that it is non-empty and allows for a measurable selection.

Then, $v = v^\star$ and any plan $\pi^\star$ generated by G achieves the maximum in (7.3).

The result can be strengthened to show that a contingent plan is optimal if and only if it is generated by G. But the details do not add to the intuition of these problems. It can also be shown to apply to the more general problem in (7.2) under slight modifications to the assumptions of the theorem.

Section 9.2 of Stokey, Lucas, and Prescott (1989) focuses on bounded problems of the type most commonly encountered in economic applications and establishes how to apply a version of the contraction mapping theorem that leads to the construction of solutions to the dynamic programing problem in (7.4) and shows that the Envelope theorem applies to these problems (under additional continuity assumptions) so as to obtain the derivatives of the value function v in therms of the payoff function F.

7.4. Markov Processes over States

Consider again the dynamic programming problem in (7.4) and suppose that its solution is a policy function G (single-valued). The objective is to describe the behavior and properties of the sequence of states $\{s_t\} = \{(x_t, z_t)\}$ as a function of the properties of the transition function of z, Q, and the policy function G. More precisely, the objective is to show that $\{s_t\}$ follows itself a Markov process with some transition function P.

The main result here is

Theorem 7.2. *Let $(X, \mathcal{X})$ and $(Z, \mathcal{Z})$ be measurable spaces with their product space $(S, \mathcal{S})$ also measurable. Let $Q : Z \times \mathcal{Z} \to [0, 1]$ be a transition function for the exogenous state z and $G : S \to X$ a policy function for the endogenous state x.*

If G is measurable with respect to $\mathcal{Z}$ then the function $P : S \times \mathcal{S} \to [0, 1]$ defined as

$$P\left((x, z), A \times B\right) = \begin{cases} Q(z, B) & \text{if } G(x, z) \in A \\ 0 & \text{if } G(x, z) \notin A \end{cases}$$

is a transition function on the state space $(S, \mathcal{S})$ defining a Markov process.

So, out of the solution of the dynamic problem comes a stochastic process for states. In this case the process is quite straightforward: the exogenous state evolves according to $Q(z, \cdot)$ and the endogenous state evolves deterministically because the decision maker has full control over it $\left(\phi\left(x, y, z^{'}\right) = y\right)$. Because the evolution of the endogenous state is deterministic it either is in a set $A \in \mathcal{X}$, or it is not. If it is, then the transition function P just needs to care about the conditional probability of $z^{'} \in B \in \mathcal{Z}$ given the current value of the exogenous state, z.

This construction of the Markov process will be crucial for the study of heterogenous agent models in part III.

We now turn to the properties of the Markov process characterized by P. Unsurprisingly, P inherits its properties from Q.

Theorem 7.3. *Consider the following conditions:*

(a) *X is a convex Borel set in $\mathbb{R}^{\ell}$ with Borel subsets $\mathcal{X}$;*

(b) *either Z is countable with $\mathcal{Z} = 2^{Z}$ its power set (set of all subsets) or Z is a compact (Borel) set in $\mathbb{R}^{k}$with Borel subsets $\mathcal{Z}$ and Q satisfies the Feller property;*

(c) *G is continuous.*

If all conditions are satisfied, then P satisfies the Feller property.

This result is important because, as seen in Section 5, the Feller property is key for most results in Markov processes that we care about. All the conditions in the theorem are used so that continuity makes sense. When Z is countable continuity is immediate on z and so Q already satisfies the Feller property. The insistence on Borel sets plays a similar role when the sets are uncountable.

The main offshoot of this Theorem is establishing the existing of an invariant distribution for the states under the solution of the dynamic programming problem. This is what we refer to as the *stationary distribution* of the states. See Theorem 5.4. When P also satisfies the Monotone property the invariant distribution is unique and the distribution of states converges to it, as in Theorem 5.5 when X and Z are rectangles in Euclidean spaces.

When the endogenous state evolves according to the law of motion $\phi\left(x, y, z^{'}\right)$ we must adjust the construction of the transition function for the states. The key difference is that $x^{'}$ is now stochastic and the is therefore a distribution over its possible values given the previous state (x, z). This distribution is constructed from G and Q similar to the case above.

To better understand the construction of the Markov process take a step back. The transition function $P((x, z), A \times B)$ is answering the question of how likely is it that

$\left(x', z'\right) \in A \times B$ given the values of (x, z). But the value of $x' = \phi\left(x, y, z'\right)$ depends on z'. In fact, z' is the only unknown given that we have (x, z). So, we can ask what is the set of values of z' that would make $x' \in A$ hold and then see if those values of z' are also in B. Call that set H and define it (in general as a correspondence) as

$$H\left((x, z), A\right) = \left\{z' \in Z \,|\, x' = \phi\left(x, G\left(x, z\right), z'\right) \in A\right\}.$$

The probability that $\left(x', z'\right) \in A \times B$ given (x, z) is then the probability that $z' \in H \cap B$ given z, so we define the transition function as

$$P\left((x, z), A \times B\right) = Q\left(z, H\left((x, z), A\right) \cap B\right).$$

Under appropriate assumptions P inherits the Feller property from Q as above.

8. Stochastic Recursive Competitive Equilibrium

We now turn to study dynamic economies subject to shocks using the tools developed above. The type of economies we are interested in are those in which individual agents interact with each other through markets. The main feature of these economies is therefore the price-taking behavior of agents. This is as opposed to models of imperfect competition or models of search friction with bilateral trading covered later in the course.

We start by describing the Neoclassical Growth Model, which constitutes the backbone of most models used in macroeconomics. The economy is populated by a representative firm and a representative household.[7] For now, there is no government. We first outline the problem of the firm and the household and then discuss how to cast them recursively, how to understand the stationary equilibrium, and how to compute the solution.

The firm produces using a constant-returns-to-scale technology that combines capital and labor. The firm chooses capital and labor to maximize its profits every period taking as given its current productivity (z_t) the period's prices: the rental rate of capital (r_t) and the wage rate (w_t). In this formulation, the problem of the firm is static:

$$\pi_t = \max_{\left\{k_t, \ell_t^d\right\}} f\left(z_t, k_t, \ell_t^d\right) - (r_t + \delta) k_t - w_t \ell_t^d.$$

The household chooses contingent plans for consumption and labor (or equivalently leisure) taking as given the return on their assets (r_t) and the wage rate (w_t). The household owns the firms and hence receives the profits the firm generates (π_t), which are also taken as given by the household. In the deterministic case, the household problem is

$$\max_{\{c_t, \ell_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(c, \ell) \qquad \text{s.t. } c_t + a_{t+1} = (1 + r_t) a_t + w_t \ell_t + \pi_t,$$

for some initial value of assets a_0. The solution of the household problem is complex because the consumption and labor plans have to be contingent on any sequence of $\{(r_t, w_t, \pi_t)\}$ that can arise. We will return to this problem later.

[7] A more formal and complete formulation of the model would introduce a continuum of agents (or households) and firms who populate the economy. The households would be price takers and their preferences would have to be homothetic. The price-taking assumption makes their constraints linear, or put another way homogeneous of degree one, the homotheticity ensures that their choices are scale free, so that the choice of an agent with half the income of another agent is to consume half as much of every good. The firms would also be price takers and operate a technology that has constant-returns-to-scale (homogeneous of degree one). This ensures that the production choices of the firms scale one-to-one. These assumptions are enough for aggregation into a representative household and a representative firm.

Equilibrium requires that markets clear along any sequence of prices, so that

$$\ell_t = \ell_t^d; \qquad a_t = k_t; \qquad c_t + a_{t+1} = f(z_t, k_t, \ell_t) + (1-\delta)\, a_t.$$

However, the variable z_t is exogenous (it is not determined by any decision maker in the economy) and random. In particular $\{z_t\}$ follows a Markov Process with transition function Q and initial value z_0. This requires some extra notation can help describe the economy. To make things simpler we assume (for now) that z takes on finitely many values. Let $z^t = (z_0, z_1, \ldots, z_t)$ be the history of shocks in the economy, taking z_0 as given, and S^t the space of all histories, and $\mu_t\left(z^t\right)$ give the probability of history z^t $\left(\sum_{z^t \in S^t} \mu\left(z^t\right) = 1\right)$. Then, the problem of the household can be formally written as

$$v(a_0) = \max_{\{a_t(\cdot), c_t(\cdot), \ell_t(\cdot)\}} \sum_{t=0}^{\infty} \sum_{z^t \in Z^t} \beta^t u\left(c_t\left(z^t\right), \ell^t\left(z^t\right)\right) \mu_t\left(z^t\right)$$

$$\text{s.t.} \sum_{t=0}^{\infty} \sum_{z^t \in Z^t} p_t\left(z^t\right)\left[\left(1 + r_t\left(z^t\right)\right) a_t\left(z^{t-1}\right) + w_t\left(z^t\right)\ell_t\left(z^t\right) + \pi_t\left(z^t\right) - c_t\left(z^t\right) - a_{t+1}\left(z^t\right)\right],$$

where we abuse notation by writing $a_0\left(z^{-1}\right) = k_0$ and $p_t\left(z^t\right)$ is the price of future resources (the stochastic discount factor).

An Arrow-Debreu equilibrium of this economy is therefore defined as sequences of functions that depend on histories of shocks. That is, an equilibrium is a set sequences for quantities $\left\{a_t(\cdot), c_t(\cdot), \ell_t(\cdot), k_t(\cdot), \ell_t^d(\cdot), \pi_t(\cdot)\right\}$ and prices $\left\{p_t(\cdot), r_t(\cdot), w_t(\cdot)\right\}$ such that

(a) Households maximize the present discounted value of utility with $\{a_t(\cdot), c_t(\cdot), \ell_t(\cdot)\}$, taking as given prices $\left\{p_t(\cdot), r_t(\cdot), w_t(\cdot)\right\}$ and transfers $\left\{\pi_t\left(s^t\right)\right\}$.

(b) Firms maximize their period profits with $\left\{k_t(\cdot), \ell_t^d(\cdot)\right\}$ taking as given prices $\{r_t(\cdot), w_t(\cdot)\}$

$$\pi_t\left(z^t\right) = f\left(z_t, k_t\left(z^t\right), \ell_t^d\left(z^t\right)\right) - \left(r_t\left(z^t\right) + \delta\right) k_t\left(z^t\right) - w_t\left(z^t\right)\ell_t^d\left(z^t\right).$$

(c) Markets clear for every history

$$\ell_t\left(z^t\right) = \ell_t^d\left(z^t\right); \qquad a_t\left(z^{t-1}\right) = k_t\left(z^t\right);$$

$$c_t\left(z^t\right) + a_{t+1}\left(z^t\right) = f\left(z_t, k_t\left(z^t\right), \ell_t\left(z^t\right)\right) + (1-\delta)\, a_t\left(z^{t-1}\right).$$

(d) The initial conditions are satisfied, so that $a_0\left(z^{-1}\right) = k_0$.

In the Arrow-Debreu equilibrium all trading happens at time 0, taking as given (k_0, z_0). We can alternatively define a sequential markets equilibrium for this economy. This definition avoids introducing the stochastic discount factor. WE instead have the equilibrium be a set sequences for quantities $\left\{a_t(\cdot), c_t(\cdot), \ell_t(\cdot), k_t(\cdot), \ell_t^d(\cdot), \pi_t(\cdot)\right\}$ and prices $\{r_t(\cdot), w_t(\cdot)\}$ such that

(a) Households maximize the present discounted value of utility with $\{a_t(\cdot), c_t(\cdot), \ell_t(\cdot)\}$, taking as given prices $\{r_t(\cdot), w_t(\cdot)\}$ and transfers $\left\{\pi_t\left(s^t\right)\right\}$.

$$v(a_0) = \max_{\{a_t(\cdot), c_t(\cdot), \ell_t(\cdot)\}} \sum_{t=0}^{\infty} \sum_{z^t \in Z^t} \beta^t u\left(c_t\left(z^t\right), \ell^t\left(z^t\right)\right) \mu_t\left(z^t\right)$$
$$\text{s.t. } \left(1 + r_t\left(z^t\right)\right) a_t\left(z^{t-1}\right) + w_t\left(z^t\right) \ell_t\left(z^t\right) + \pi_t\left(z^t\right) \geq c_t\left(z^t\right) a_{t+1}\left(z^t\right) \qquad \forall_t \forall_{z^t \in Z^t},$$

(b) Firms maximize their period profits with $\left\{k_t(\cdot), \ell_t^d(\cdot)\right\}$ taking as given prices $\{r_t(\cdot), w_t(\cdot)\}$

$$\pi_t\left(z^t\right) = f\left(z_t, k_t\left(z^t\right), \ell_t^d\left(z^t\right)\right) - \left(r_t\left(z^t\right) + \delta\right) k_t\left(z^t\right) - w_t\left(z^t\right) \ell_t^d\left(z^t\right).$$

(c) Markets clear for every history

$$\ell_t\left(z^t\right) = \ell_t^d\left(z^t\right); \qquad a_t\left(z^{t-1}\right) = k_t\left(z^t\right);$$

$$c_t\left(z^t\right) + a_{t+1}\left(z^t\right) = f\left(z_t, k_t\left(z^t\right), \ell_t\left(z^t\right)\right) + (1 - \delta) a_t\left(z^{t-1}\right).$$

(d) The initial conditions are satisfied, so that $a_0\left(z^{-1}\right) = k_0$.

The sequential problem of the agents in this economy must therefore keep track of an impossible large state vector because the optimal choices depend on its complete history. Tackling this problem proves to be impractical if not impossible. Because of that we seek to re-formulate the household's problem in its recursive form. Doing so will also provide us with a definition of recursive competitive equilibrium (RCE).

8.1. Recursive Competitive Equilibrium

The objective now is to cast the problem recursively. This allows us to provide a clearer definition of equilibrium and (later) to solve the problem using the tools of dynamic programming developed above and the computational tools introduced below.

The main question of the recursive problem is what constitute the relevant state of the economy. Crucially, the household is a price taker: they have no clue about the aggregate effect of their choices. Equivalently you can think of the household as atomistic, so that they know that their (individual) actions have no effect on the aggregates of the economy. However, the states must provide enough information to solve the problem and to forecast how the states themselves evolve.

In general we have $s_t = (a_t, z_t, \Gamma_t)$ be the state of the individual representative household, it includes the household's capital (a), the aggregate productivity (z), and the distribution of states in the economy (Γ). Keeping track of the distribution of states is in principle necessary in order to compute prices. The sequential problem circumvents this problem by making all variables (prices included) depend on the full history of exogenous shocks. That, of course, contains all the information necessary. However, the recursive problem cannot depend the history of states, $s^t = (s_0, s_1, \ldots, s_t)$.

To solve this issue we use the structure of the economy. The key is that the economy we are studying can be aggregated. This means that the underlying distribution of households does not matter and that only the aggregate (average) capital is relevant: $K_t = \int a_t d\Gamma_t$. Because of this, an individual household only needs to keep track of aggregate capital and not of the whole distribution. Crucially, knowing the aggregate capital is enough to compute the relevant prices in the economy. This makes the state $s_t = (a, z, K)$.[8]

The household's recursive problem is then

$$
\begin{aligned}
V(\underbrace{a}_{\text{Ind. State}}, \underbrace{z, K}_{\text{Agg. States}}) = \max_{\{c,\ell,a'\}} \; & u(c,\ell) + \beta \int_{z'} V\left(a', z', K'\right) Q\left(z, dz'\right) \\
\text{s.t.} \quad & c + a' = (1 + R(z,K))\, a + W(z,K)\, \ell + \Pi(z,K); \\
& K' = G_k(z,K).
\end{aligned}
$$

This problem looks a lot like the dynamic programming problems we h ave discussed before, with the exception that it has to include functions that map the aggregate states into prices, $r = R(z, K)$ and $w = W(z, K)$, and profits, $\Pi(z, K)$, and that incorporate the evolution of the aggregate state, $G_k(z, K)$. These functions are taken as given by the household and are found as part of the equilibrium defined below.

An Recursive Competitive Equilibrium (RCE) is a set of a value function V, policy functions g_k and g_ℓ, updating functions G_k and price and profit functions R, W, and Π, such

[8]The state is often written as (k, z, K) emphasizing the difference between the "little k" faced by an individual household and the "big K" faced by the economy as a whole.

that:[9]

(a) The value function V and policy functions g_a and g_ℓ solve the household's dynamic programming problem, taking as given the updating and price functions G_k, R, W, and Π.

(b) The firms maximize profits taking prices as given. This implies that the pricing functions R and W satisfy the firm's first order conditions

$$R(z,K) = f_k(z,K,G_\ell(z,K)) - \delta \qquad W(z,K) = f_\ell(z,K,G_\ell(z,K)),$$

where $L = G_\ell(z,K)$ gives the aggregate equilibrium level of labor as a function of the aggregate states. And profits satisfy

$$\Pi(z,K) = f(z,K,G_\ell(z,K)) - (R(z,K)+\delta)K - W(z,K)G_\ell(z,K).$$

Note: The price and profit functions are evaluated at the equilibrium levels of capital, K, and labor, $L = G_\ell(z,K)$, and not at the firms' capital and labor demand. This already imposes market clearing for the capital and labor markets because the aggregate variables are consistent with households' supply of assets and labor, as noted below. Walras' law then implies that the good's market clears.

(c) Updating functions G_k and G_ℓ are consistent with individual optimization

$$G_k(z,K) = g_a(K,z,K);$$
$$G_\ell(z,K) = g_\ell(K,z,K).$$

Note: This consistency condition plays two roles. First, it acknowledges that the household is a representative agent, so that, even though the household acts individually without having any effect on aggregates, the household assets a are equal to the aggregate capital K (from the point of the view of the economy, but not of an individual household). This is captured by evaluating the policy functions of the household in (K,z,K) and by making the evolution equation for aggregate capital G_k

[9] The definition of an RCE can be equivalently be done in terms of stochastic processes for the equilibrium variables (capital, labor, prices, etc.). That is, the equilibrium is the processes for quantities and prices. These stochastic processes are the equivalent of the sequences of quantities and prices that define a non-stochastic equilibrium. However, defining the equilibrium in that way is cumbersome. The stochastic processes are all constructed from the underlying Markov process for z and the policy and price functions that define the RCE. The construction is carried out as shown in Section 7.4.

consistent with household savings g_a. Second, it acknowledges market clearing by making the aggregate equilibrium labor $L = G_\ell(z, K)$ be consistent with household labor supply, g_ℓ.

Crucially, consistency only has to apply in equilibrium. This allows us device an algorithm to solve for the RCE. The key is that consistency does not have to hold as you converge to the equilibrium because the household dynamic problem can be solved given any update functions.

Algorithm

input : Guess for updating functions (G_k, G_ℓ)

output : $V, g_k, g_\ell, G_k, G_\ell$

1. Solve the DP problem of the agent given G_k, G_ℓ:
   $(V, g_k, g_\ell) = T(V; G_k, G_\ell)$ (a fixed point problem) ;
2. Update updating functions:
   $G_k(z, K) = g_k(K, z, K) \qquad G_\ell(z, K) = g_\ell(K, z, K)$;
3. Check convergence in updating functions ;
4. Repeat (1)-(3) until convergence ;

Algorithm 1: RCE Algorithm

Solving the problem requires solving the fixed point characterizing the solution to the household's Bellman equation. Unfortunately, this implies that the curse of dimensionality applies because you have to solve the agent's problem off-equilibrium. That is, you need to know $g_k(a, z, K)$ for any combination of (a, K), even though in equilibrium $a = K$. For efficient economies, where the first welfare theorem applies, we can avoid this problem by focusing on the planner's problem and then constructing the equilibrium prices. However, most applications involve economies with market failures, or distortions (like taxes!) that prevent us from doing this.

8.2. Stationary Equilibrium: What does it mean?

We now discuss a key property of the equilibrium. When the process for the exogenous shocks is stationary and the problem of the firm and the household satisfy certain regularity conditions, the equilibrium converges to a stationary equilibrium. The objective of this subsection is to discuss what that means.

First we discuss informally what the regularity conditions are. What we want is to establish conditions that produce "well-behaved" policy functions for the endogenous states (capital in this case), that is, continuous and monotone. The problem must also

satisfy standard Inada and transversality conditions that guarantee that it is effectively bounded.

Continuity is inherited from the continuity of payoff functions (in this case u and f) and the properties of the feasible correspondence. See the Theorem of the Maximum for more on this. It also requires that the Markov Process for z satisfies the Feller property (because of the expectation in the value function).

Monotonicity requires having a sense of what is "better" in the context of the problem. We interpret z as productivity and hence it makes sense to interpret higher values of z as being better. For the resulting solution to be monotone in the states (k, z, K) we need the payoffs to be monotone (as we usually assume), the transition function Q to be monotone (see Definition 5.4 and Proposition 5.5).

Crucially, the conditions imposed over the Markov process for z already guarantee that it is a stationary process with a unique invariant distribution to which it converges, regardless of its initial condition, z_0. The question is whether the stochastic process followed by the equilibrium variables (capital) are also stationary and converge to an invariant distribution. The stationary equilibrium is then just a recursive competitive equilibrium for which the stochastic process of the endogenous variables is stationary.

The concept of the stationary equilibrium is immediate in the non-stochastic case. Then, policy functions map (deterministically) a value of the endogenous state (capital) into a new value for itself. A sequential markets equilibrium and a recursive competitive equilibrium are defined just as in the previous subsection, except that they do not depend on the history of shocks (as there are none). The equivalent of being stationary and having an invariant distribution is then to have a single steady-state value that satisfies the equilibrium conditions. Intuitively, the deterministic case is like the stochastic one with degenerate distributions, so the invariant degenerate distribution puts full probability on a single value of the variable. That is the steady state.

In the stochastic case, the equilibrium is composed by functions that map the realization of the stochastic process for productivity (z) into values for quantities (K) and prices (r, w). When these functions are measurable with respect to the underlying productivity process they form themselves a stochastic processes (the sequence of the random variables for quantities and prices). The construction of these stochastic processes is carried out as in Section 7.4. Then, the stationary equilibrium is an endogenously determined probability distribution over the state variables, with the properties of a Markov chain induced by the policy functions and the exogenous process for shocks.

8.3. Profit Maximization: A note

Remark. This sub-section reproduces lessons taught by Tim Kehoe at the University of Minnesota.

We often assume that the production function has constant returns to scale, that is, that it is homogenous of degree 1. This is in part for making possible the aggregation of the economy from individual competitive firms having the same behavior (at scale) than a representative price-taking firm. This assumption has important consequences for how we set up the firms' problem. The objective of this note is to make these consequences apparent. Before tackling them, it is useful to state Euler's theorem for homogeneous functions.

Definition 8.1. (Homogeneous Function) A function $f : \mathbb{R}^N \to \mathbb{R}$ is homogeneous of degree $g > 0$ if $f\left(\lambda \cdot \vec{x}\right) = \lambda^g \cdot f\left(\vec{x}\right)$ for $\lambda > 0$.

Theorem 8.1. ***(Euler's Theorem for Homogeneous Functions)*** *Let $f : \mathbb{R}^N \to \mathbb{R}$ be differentiable function homogeneous of degree g. Then $\frac{\partial f(x)}{\partial x_i}$ is homogenous of degree $g - 1$ for all $i \in \{1, \ldots, N\}$ and*

$$g\, f\,(x) = \sum_{i=1}^{N} \frac{\partial f\,(x)}{\partial x_i} x_i.$$

There are three main implications of Euler's theorem for the equilibria of macroeconomic models with constant returns to scale production functions.

1) Profit maximization is ill-defined. Consider the profits of a price-taking firm that produces renting capital and hiring labor with a technology described by $Y = zF\left(K^d, L^d\right)$ that has constant returns to scale, so that $F\left(\lambda K^d, \lambda L^d\right) = \lambda F\left(K^d, L^d\right)$ for $\lambda > 0$. The profits of the firm, π, also have constant returns to scale. To see this let

$$\pi\left(K^d, L^d\right) = zF\left(K^d, L^d\right) - (r + \delta)\, K^d - wL^d$$

and consider an increase in the scale of the firm by a factor of $\lambda > 0$. It is immediate that

$$\pi\left(\lambda K^d, \lambda L^d\right) = zF\left(\lambda K^d, \lambda L^d\right) - (r + \delta)\, \lambda K^d - w\lambda L^d = \lambda \pi\left(K^d, L^d\right).$$

This has an important implication for firm behavior: If the firm makes a profit when demanding K^d units of capital and L^d units of labor, $\pi\left(K^d, L^d\right) > 0$, then it can scale up that profit by scaling up that demand. Hence the optimal scale of the firm, and hence it demand for inputs, would be ill-defined (it would be infinite). If the firm makes a loss for

all $\left(K^d, L^d\right)$, $\pi\left(K^d, L^d\right) < 0$, then it is optimal to set $K^d = L^d = 0$ and not produce. Finally there is a knife edge case where $\pi\left(K^d, L^d\right) = 0$ for all $\left(K^d, L^d\right)$, but the firm's scale and demand for inputs is still ill-defined because the firm is indifferent between any scale of production.

This makes the profit maximization problem ill-defined because the answer to

$$\max_{\left\{K^d, L^d\right\}} \pi\left(K^d, L^d\right)$$

is either 0 or infinity and there is no optimal scale for the firm (other than the case in which the firm does not operate) .

2) Equilibrium profits must be zero. Despite the profit maximization problem being ill-defined it does tell us that the only equilibrium outcome arises in the knife edge case of zero profits. This is has to be the case for the firm to operate but have a finite scale, and hence a finite demand for inputs.

Turns out that this is precisely the case that arises when prices reflect the marginal product of inputs. The reason lies in Euler's theorem. The constant returns to scale technology implies that

$$\begin{aligned}
\pi\left(K^d, L^d\right) &= zF\left(K^d, L^d\right) - (r+\delta)K^d - wL^d \\
&= zF\left(K^d, L^d\right) - zF_K\left(K^d, L^d\right)K^d - zF_L\left(K^d, L^d\right)L^d \\
&= zF\left(K^d, L^d\right) - zF\left(K^d, L^d\right) \\
&= 0.
\end{aligned}$$

3) Firms cost minimize. The condition for zero profits above is true only if the prices coincide with the marginal product of inputs given the firms' input demand. How can we guarantee this coincidence? The answer lies in the firms' cost minimization problem. Regardless of the scale of the firm, the firm will always minimize its cost given that scale. So it holds true that

$$C(Y) = \min_{\left\{K^d, L^d\right\}} (r+\delta)K^d - wL^d \qquad \text{s.t. } zF\left(K^d, L^d\right) \leq Y$$

The first order conditions of this problem give us

$$(r+\delta) = \mu z F_K\left(K^d, L^d\right) \qquad w = \mu z F_L\left(K^d, L^d\right),$$

where μ is the Lagrange multiplier of the constraint that we can (later) normalize to one.

Euler's theorem helps again. The marginal products are homogeneous of degree zero because the production function is homogeneous of degree 1. Put another way, the marginal products are scale-invariant. That means that we can write

$$(r+\delta) = \mu z F_K\left(\frac{K^d}{L^d}, 1\right) \qquad w = \mu z F_L\left(\frac{K^d}{L^d}, 1\right).$$

This has two important implications. First, the marginal products of inputs depend only on their ratio and not on the their individual scale. So, the condition for zero profits only requires that the firm input demand has the same ratio as the one implied by prices. Second, the optimal ratio of input demand is determined by the ratio of prices, implicitly by the following equation:

$$\frac{r+\delta}{w} = \frac{F_K\left(\frac{K^d}{L^d}, 1\right)}{F_L\left(\frac{K^d}{L^d}, 1\right)}.$$

This is actually crucial, because it guarantees that, given prices r and w, the firm will optimally choose the precise ratio of inputs that makes zero profits hold.

It is left to determine prices and the scale of production. This is done in equilibrium. Market clearing demands that the level of output is consistent with the aggregate demand in the economy (consumption, savings, government spending, etc.) and that the labor and capital used in production are the same as the assets and labor supplied by households. Then, the same equation of the ratio of prices and inputs used to determine the optimal demand of inputs given prices, is used in the inverse manner to determine the equilibrium ratio of prices given the supply of assets and labor. Similarly, the scale of production is obtained by agree An equilibrium allocation equates the demand and supply of capital and labor. The scale is then provided given Y coming from aggregate demand.

The bottom line is that, in equilibrium, the ratio of prices and the ratio of inputs are determined by technology and the scale of production is demand-determined.

This can be seen in the definition of the RCE at the beginning of this section where prices were obtained as

$$R(z,K) = f_k(z,K,G_\ell(z,K)) - \delta \qquad W(z,K) = f_\ell(z,K,G_\ell(z,K)),$$

with K being the aggregate assets in the economy (a state) and $L = G_\ell(z,K)$ the aggregate supply of labor in the economy (consistent with household optimization).

8.4. Computing the Equilibrium: Value function iteration (the discrete case)

The objective is now to illustrate how to compute the equilibrium. We take advantage of the fact that the the economy we described is efficient and that we can therefore use the planner's problem to construct all the equilibrium functions. We further simplify the problem by getting rid of the labor choice. This simplifies the exposition. The planner's problem is to choose aggregate quantities subject to feasibility:

$$V(K,z) = \max_{\{C,K\}} u(C) + \beta \int_{z'} V\left(K', z'\right) Q\left(z, dz'\right)$$
$$\text{s.t.} \qquad C + K' = f(z,K) + (1-\delta)K.$$

The solution to the planner's problem immediately gives us the equilibrium for individual quantities $\{c, a, k\}$ and prices $\{r\}$ by setting

$$c = C \qquad k = a = K \qquad r = f_k(z,K,L) - \delta.$$

We can solve the problem using value function iteration. The key is that, unlike the household problem above, we do not need to condition on the functions for prices or aggregates.

Algorithm

Result: Fixed Point of Bellman Operator T

$n = 0$; $V^0 \in S$; $dist_V = 1$;
while $n \leq N$ & $dist_V > tol_V$ **do**
  $V^{n+1} = TV^n$;
  $dist_V = d\left(V^{n+1}, V^n\right)$;
end
if $dist_V \leq tol_V$ **then**
  Obtain g from TV^n;
else
  You are in trouble... something went wrong;
end

Algorithm 2: Value Function Iteration

While actually solving the problem as posed can be challenging (because of the difficulties in making continuous choices and taking expectations) it is possible to approximate it with a related (and much simpler problem) in which the whole problem is discretized. This is the simplest implementation of value function iteration. The key

advantage is that there are no continuous choices (or integral), instead, the problem consists in choosing the best value of capital from a pre-specified and fixed grid (hence its common name of grid search).

The approximation of the (continuous) dynamic programming problem with a discrete one does not require the use of derivatives and is robust to complications such as kinks in the choice set, or asymmetries in the functions being used. It is also very easy to implement. However, it is not (in general) a very precise approximation, and it has a low rate of convergence, making it slow. This problem is compounded by the curse of dimensionality, which bites particularly hard for discrete problems because they require large state spaces in order to improve the accuracy of the approximation (more on how to gauge accuracy at the end of this section).

The discrete problem is

$$V\left(k_i, z_j\right) = \max_{k' \in \{k_1, \ldots, k_I\}} U\left(z_j k_i^{\alpha} + (1-\delta) k_i - k'\right) + \beta \sum_{j'=1}^{J} \Pi_{jj'} V\left(k', z_{j'}\right)$$

where we have replaced the constraint, leaving consumption implicitly defined by the choice of capital which is now discrete. Conveniently, everything in the problem is now a vector or a matrix:

$$V = \left[V_{ij}\right]; \qquad \vec{k} = \left[k_1, \ldots, k_I\right]^T; \qquad \vec{z} = \left[z_1, \ldots, z_J\right]^T; \qquad \Pi = \left[\Pi_{jj'} = Q\left(z_j, \left\{z_{j'}\right\}\right)\right];$$

$$U = \left[U_{ijh} = u\left(z_j k_i^{\alpha} \qquad \vec{k} = \left[k_1, \ldots, k_I\right]^T - k_h'\right)\right];$$

This allows us to solve the problem of choosing $k' \in \{k_1, \ldots, k_I\}$ for every pair of $\left(k_i, z_j\right)$ in two different ways. Either looping through all the pairs of states, or collapsing the matrix of payoffs along its third dimension.

This algorithm can be sped up in many programming languages by operating directly on matrices, instead of relying on loops. This also leads to a more concise program.

In order to evaluate the accuracy of the solution we make use of **Euler Residuals**. These are the residuals in the first order conditions of the actual problem, which should be zero

Algorithm

Function `T(`*V_old,k_grid,z_grid,*α, β`)`**:**

n_k = length(k_grid)

n_z = length(z_grid)

V = zeros(n_k, n_z); G_{kp} = zeros(n_k, n_z); G_c = zeros(n_k, n_z)

for i = *1:*n_k **do**

  for j = *1:*n_z **do**

    V_aux = zeros(n_k)

    for h = *1:*n_k **do**

      V_aux[h] = u(k_grid[i],z_grid[j],k_grid[h];α, β) + βsum($\Pi_{j,j'}$V_old[h,j'])

    end

    V[i,j], G_{kp}[i,j] = findmax(V_aux)

    G_c[i,j] = f(k_grid[i],z_grid[j]) + (1-δ)k_grid[i] - k_grid[G_{kp}[i,j]]

  end

end

return V, G_{kp}, G_c

Algorithm 3: Bellman Operator: Discrete grid with loops

for the correct solution.

$$\textbf{Res}\left(k, z_j | g\right) = \underbrace{\frac{\beta \sum_{j'=1}^{J} \Pi_{jj'} U'\left(f\left(z_{j'}, g\left(k, z_j\right)\right) + (1-\delta)k - g\left(g\left(k, z_j\right), z_{j'}\right)\right) \frac{\partial f\left(z_{j'}, g\left(k, z_j\right)\right)}{dk}}{U'\left(f\left(z_j, k\right) + (1-\delta)k - g\left(k, z_j\right)\right)} - 1}_{\text{\% Error in Euler Equation}}$$

We can evaluate these residuals for values of capital in the grid used to solve the problem. The Euler residuals can help diagnose if there are parts of the state space that need to be denser (say having more grid points near low-levels of capital where the curvature of the problem is higher) or whether the approximation to the solution is satisfactory in general.

Having approximated the solution to the dynamic programming problem we can construct a Markov process for the states in the economy and obtain their stationary distribution. In the special case of discrete grid search, this is facilitated by the fact that the choice of future capital is always in the grid. We can then construct a Markov transition matrix for the state vector of the economy. In this case, the state is $s = (k, z)$, and the state space can be express as a long vector

$$\vec{s} = \left((k_1, z_1), (k_1, z_2), \ldots, (k_1, z_J), (k_2, z_1), \ldots, (k_2, z_J), \ldots, (k_I, z_1), \ldots, (k_I, z_J)\right).$$

The Markov transition matrix is therefore a square matrix with $I \times J$ rows and columns. We can build the matrix following the steps in Section 7.4. The key is that z evolves independently following the transition matrix Π, while k evolves deterministically. The matrix is then

$$\Psi_{\left(ij, i'j'\right)} = \Pi_{jj'} \chi_{\left\{i' = G_{kp}(i,j)\right\}}.$$

The properties of the stochastic process for capital and productivity then follow from this matrix. For instance, the stationary distribution is obtained from the eigenvector associated with the matrix unit eigenvalue.

8.5. Recursive Competitive Equilibrium Example: Sovereign Default

Sovereign default models form a large literature on international economics and are also a great example of stochastic dynamic programming. The choice to default required dynamics in order to have an opportunity and a reason to borrow and to introduce a relevant tradeoff around default. The model must also be stochastic in order to induce the situations in which a decision maker borrows and then finds themselves in a situation

where they opt to default. These models are also inherently inefficient, preventing the use of the planner's problem. The reason is that markets must be incomplete in order for the decision maker not to be able to fully insure against risk. Further, the borrowing and default decisions depend on prices, which are taken as given by the decision maker, but that respond endogenously (in equilibrium) to the decision maker's choices.

The basic sovereign default model follows Arellano (2008). It is a stochastic endowment economy. Output (or income) follows an exogenous (discrete) Markov process described by an underlying state $s \in \{s_1, \ldots, s_N\} = S$. The decision maker, say the government, chooses borrowing/saving and whether to default on debt. The decision is made taking as given a price schedule for debt (q) that depends on the state of the economy and the debt of the government.

The dynamic programming problem is then split in two. First there is the (discrete) choice of whether to default. The state of the government is the pair (s, b), where s is the exogenous state of the economy and b is the level of outstanding bonds to be paid to the government (so $b > 0$ means savings and $b < 0$ means debt). If the government pays they get to access the lending markets and gets a value of $V(s, b)$ but if it defaults it is thrown into financial autarky and gets a value $V^A(s)$ (that no longer depends on b because there is no debt and no access no markets). The value of the government, $V^\star$, reflects the upper envelope of this choice,

$$V^\star(s, b) = \max_{d \in \{0,1\}} \left\{ (1 - d(s, b)) V(s, b) + d(s, b) V^A(s) \right\}.$$

The value of having access to the financial markets is

$$\begin{aligned} V(s, b) = \max_{\{c, b'\}} & \left\{ \frac{c(s, b)^{1-\sigma} - 1}{1 - \sigma} + \beta \sum_{s' \in S} \pi\left(s, s'\right) V^\star\left(s', b'\right) \right\} \\ \text{s.t.} \quad & c(s, b) \leq y(s) + b - q(s, b) b'(s, b) \\ & -B \leq b'(s, b) \qquad \text{[B: borrowing limit]} \\ & 0 \leq c(s, b). \end{aligned}$$

The value of going into autarky is

$$V^A(s) = \frac{(h(y(s)))^{1-\sigma} - 1}{1 - \sigma} + \beta \sum_{s' \in S} \pi(s, s') \left(\theta V^\star(s', 0) + (1 - \theta) V^A(s') \right),$$

where $h(y) < y$ is a function that penalizes output, representing the costs of autarky.

There is no choice for the government as there are no markets that allow it to smooth consumption. The government returns to the markets with probability $\theta \geq 0$.

There is also a sector of financial intermediaries that operate in perfect competition. They are risk neutral and lend in an actuarially fair manner, meaning that their prices reflect the expected costs of default and so they break even in expectation. The profits of one of these financial intermediaries are

$$\text{Pr} = qb' - \frac{1-\delta}{1+r}b',$$

where δ is the (endogenously determined) probability of default taken as given by the intermediary. This probability comes, in equilibrium, from the optimal default choice of the government, $g^D(s, b)$, and satisfies

$$\delta = E_{s'}\left[g^D\left(s', b'\right)|s\right] \qquad \text{where } g^D\left(s', b'\right) = \begin{cases} 1 & \text{if default} \\ 0 & \text{if no default} \end{cases}.$$

Free entry gives the zero profit (break even) condition that Pr = 0 and so the debt price is (in equilibrium):

$$q\left(s, b'\right) = \begin{cases} \frac{1-\sum_{s' \in S} \pi(s') g^D(s', b')}{R} & \text{if } b' < 0 \\ \frac{1}{R} & \text{if } b' \geq 0 \end{cases}$$

A **Recursive Competitive Equilibrium** is then a set of value functions $\left\{V^\star, V^A, V\right\}$, policy functions $\left\{g^c, g^b, g^D\right\}$, and a price functional $\left\{q\left(s, b'\right)\right\}$ such that

(a) The value functions solve the Bellman equations of the government and the policy functions achieve the maximum in those equations taking the price q as given.

(b) The price q satisfies the zero profit or break even condition of the financial intermediaries

Part III

Heterogeneous Agent Models

We now want to consider economies with a continuum of agents where heterogeneity plays a role. In the models of the previous questions preferences and technology were such that we could aggregate the behavior of agents into a representative household and a representative firm. In sum, only the aggregate levels of variables like capital (wealth) mattered, and not their distribution across individuals. This abstraction is appropriate for many questions, especially those about the behavior of aggregates and business cycles, but it comes at big costs because it essentially collapses various dimensions of relevant heterogeneity into the behavior of aggregates.

The alternative to representative-agent macroeconomics is what Benjamin Moll calls *Distributional Macroeconomics*. I encourage you to checkout his lecture notes on this available here: Distributional Macroeconomics Notes. Moll's notes frame the development of macroeconomics towards a state where inequality at the micro-level matters for the behavior of aggregates, and the behavior of aggregates matters for individual agents in different ways.

We can think of two types of heterogeneity to include in our models*:*

Ex-ante heterogeneity where agents have different types that determine their possible actions, preferences, or technology. A good example of this is models of workers and entrepreneurs, where some agents are (always) workers while other agents are (always) entrepreneurs. There can also be further differences within each type of agent, for example some entrepreneurs can have permanently high- or low-productivity. See Guvenen, Kambourov, Kuruscu, and Ocampo (2023a) for an application of these ideas.

Ex-post heterogeneity where the differences among agents arise from their endogenous reaction to the realization of idiosyncratic and aggregate shocks they face. The differences among agents are then seen in their states. Some are in debt, some have savings; some have high income or human capital, some have low income or human capital; some own housing, some rent (and some houses are larger than others).

Whether the model economy exhibits ex-ante or ex-post heterogeneity (or both), agents are subject to shocks to their ability to generate income (say unemployment and employment as in the original work of İmrohoroğlu 1992), or to their age (some are young, some are old), or to their health, etc. The distribution of these shocks along with the endogenous response of the different agents leads to an endogenous distribution across states. In the absence of aggregate shocks, this distribution converges to a stationary

distribution of agents. That will be the main objective of our theory: develop models of the distribution of agents across states.

Why do we care so much about the distribution of agents across states? The distribution is what connects the micro-behavior of agents with the macro-aggregates of the economy. The distribution is what allows us to explicitly aggregate individual behavior in our models. We will devote most of our time to models where the distribution does not change in the long run because there are no aggregate shocks. In these models, the lives of individuals are always changing as they face shocks and respond to them, moving through the distribution, but the distribution itself is always the same because the changes in individuals' lives are all uncoordinated.

In terms of the machinery we have developed so far, the stationary distribution does not describe the behavior of a single random variable across time (as it follows a stationary stochastic process) but the cross-sectional behavior of a population. In this context, aggregates are of course constant (as in the definition of a steady state) because they only depend on the distribution, and not on the actions of any individual.

There is a final ingredient of these models. Critical to their working is the notion of market incompleteness. If markets were complete, agents could perfectly insure the idiosyncratic risk they face and there would be no differences in allocations across them beyond those explained by preferences and technology. If all heterogeneity is ex-post heterogeneity, complete markets would effectively do away with these ex-post differences. The most common form of market incompleteness is the absence of state-contingent bonds. Instead, agents can only trade in bonds (or assets) that pay the same regardless of the realization of idiosyncratic shocks. This prevents agents from fully insuring against income fluctuations, for instance.

9. The Income Fluctuations Model and Precautionary Savings

The basic form of the heterogeneous agent model is built to generate ex-post household heterogeneity and to be consistent (as much as possible) with the neoclassical growth model. However, unlike the neoclassical growth model, there are no aggregate shocks.

The economy is populated by a continuum of households who are ex-ante identical. That means that they have the same preferences and the solve the same problem. These households are subject to idiosyncratic income shocks. These shocks are most often interpreted as variation in the household's labor-income. The households have access to a savings technology in the form of assets (or bonds) that pay the same return r in all states (corresponding to the value of the income shocks of the households). Households are

further subject to a borrowing constraint, so that their (net) assets have to be above a given lower bound $\underline{a}$.

The problem of the household is (by construction) almost identical to the household problem in the Recursive Competitive Equilibrium of the neoclassical growth problem:

$$V(\epsilon, a) = \max_{\{c,\ell,a'\}} u(c, \ell) + \beta E\left[V\left(\epsilon', a'\right)|\epsilon\right]$$
$$\text{s.t. } c + a' = (1 + r)\, a + w f(\epsilon, \ell) \qquad a' \geq \underline{a}$$

In this problem the household is subject to shocks ϵ to their earning ability, as before we assume that ϵ follows a Markov process with transition function Q. The household can, in principle, choose how much to work but how ϵ determines how labor is translated into earnings. Savings take the form of assets a.

We are already setting the problem in its stationary form, as evidenced by the fact that prices (r, w) are constant. We will formalize this idea later when we provide the notion of equilibrium.

Precautionary Savings and the Rate of Return. An important aspect of this problem is that, unlike that of the planner in the neoclassical growth model, the returns to assets are constant. That is, there is no curvature in the budget constraint. This is of course a consequence of the household being a price taker. Tied to this fact is the (newly introduced) disconnect between the individual's income and the return on assets. In the representative agent economy we discussed above, wages and returns moved in response to the same (aggregate) conditions. Here, each household's income moves in response to idiosyncratic factors (captured by ϵ) with prices held constant (and equal for all households).

These aspects matter because they introduce a new motive for savings that has been termed "*precautionary savings*." To understand this motive for savings we start by contrasting the heterogeneous agent model we introduce here with the neoclassical growth model and its representative agent to then contrast them with what is happening in the new model.

Savings and returns in a representative agent world Savings in the neoclassical growth model are determined by the inter-temporal trade-off of spending resources in the present in order to receive returns in the future. These returns corresponded to the marginal product of the capital being accumulated through savings. The representative

household's Euler equation is

$$u_c(C, L) = (1 + r)\,\beta E\left[u_c\left(C', L'\right)\right].$$

In steady state with no aggregate fluctuations (the corresponding version of the stationary recursive competitive equilibrium we will study below) we have the standard result that the return on assets is equal to the (inverse of the) inter-temporal rate of discount

$$1 + r = 1/\beta.$$

This is a no-arbitrage expression. Without any risk (exogenous variation in future values of consumption and leisure), the household is indifferent between saving an extra unit of goods or not. In other words, because $(1 + r)\,\beta = 1$ the discounted value of the returns is the same as the present cost of the goods being saved. This indifference condition prevents the (representative) household from engaging in arbitrage. For instance, if $1 + r < 1/\beta$ the household would want to borrow infinitely. The linearity (lack of curvature) of the household's budget introduces the same forced discussed before in the context of a firm with constant returns to scale technology. In the same way, the equilibrium level of capital (wealth or assets) is not determined by supply (which is perfectly elastic given the no-arbitrage condition above) but demand-determined. The curvature that sets the steady state level of capital comes from the firm's capital demand, that requires that

$$1 + r = f_k(z, K, L) + 1 - \delta.$$

Together these conditions provide the standard steady state condition for the neoclassical growth model

$$f_k(z, K, L) - \delta = \frac{1}{\beta} - 1,$$

equating the steady state marginal product of capital with the rate of inter-temporal discount.

Beyond determining the level of aggregates in steady state, the results above have an important economic interpretation. The sole determinant of the level of capital in the economy is the curvature of the production function (that determines the marginal productivity of capital) relative to the household's rate of discount. The household's saving motive is one of future use of capital. That is, capital is accumulated to be used. This logic is transformed in the heterogeneous agent model as we see below.

Idiosyncratic risk and the stationary rate of return The introduction of *idiosyncratic and uninsurable risk* to the model (via income fluctuations and incomplete markets) results in a new saving motive for households. They want to have savings "just in case something happens." Under this logic, assets are not accumulated with the objective of using them (say for production) but rather with the object of not using them! Savings, even in non-state-contingent assets provide a form of self-insurance against idiosyncratic income fluctuations. Households with currently high incomes (relative to their long-run average) accumulate assets in anticipation of potentially low incomes to come (they save hoping that those savings won't be needed, but knowing that they will). Households with low income can partially insure themselves by drawing on their savings. Here is where the borrowing constraint bites, as it prevents better insurance by limiting the debt that households can accumulate. This turns out to be a crucial factor determining the equilibrium aggregates in the economy.

The households' Euler equation is

$$u_c\left(c_i, \ell_i\right) = (1+r)\,\beta E_{\epsilon_i'}\left[u_c\left(c_i', \ell_i'\right)|\epsilon_i\right].$$

Even though prices and aggregates do not change, the expectation over future idiosyncratic shocks is still relevant. Moreover, because households are heterogeneous (with different asset and income levels), the Euler equation necessarily implies that some households save while other dissave, depending on the value of their current and expected consumption. Unlike the normal savings motives explained above, the household wants to save because their current income is higher than their expected future income, this will happen for some household regardless of the relationship between the rate of return and the rate of discount. Of course, more households are willing to save if there is a higher rate of return or a higher rate of discount, and vice-versa.

What happens then with the relationship between the rate of return and the rate of discount in equilibrium? Aiyagari (1994) shows that it must be that case that

$$1 + r < 1/\beta.$$

The reason is a combination of the fact that households are heterogeneous (facing idiosyncratic labor income risk) and that the presence of the borrowing constraint that introduces a key asymmetry to the problem. It cannot be that $1 + r > 1/\beta$ because then the Euler equation would imply that consumption must have an upward drift (that is, that it must be expected to increase). This means that households want to save always. Because households are infinitely lived, this implies that assets (and consumption) tend to infinity.

This cannot be a solution. See Aiyagari (1994, footnote 20) for a formal argument. The same thing occurs if $1 + r = 1/\beta$. The reason lies in the desire for self-insurance as the household wants to have constant (expected) marginal utility over time. Once again, because the household is infinitely lived, they face the risk of an arbitrarily long sequence of low-income shocks. In order to insure against that, the household needs an arbitrarily large stock of savings. This is feasible because, with $1 + r = 1/\beta$, it is costless for the household to transform current consumption into savings.

Therefore, it must be that $1 + r < 1/\beta$ in equilibrium. Under complete markets this would lead the household to accumulate infinite debt. The borrowing constraint prevents this from happening. Instead, there is a positive mass of households who de-accumulate assets and hit the constraint (following a series of low income shocks), while other households accumulate assets when they have higher levels of income. The properties of the optimal saving function in this income-fluctuation problem are explored in Huggett (1993, Theorem I), Aiyagari (1994, Sec. III, The Individual's Problem), and more explicitly in Achdou, Han, Lasry, Lions, and Moll (2022).

What does this imply for the aggregate level of assets (or capital) in the economy? Huggett (1993) and Aiyagari (1994) show that the level of assets in the incomplete-markets/heterogeneous-agent economy must be larger than in its complete-markets/representative agent counterpart. The additional assets (relative to complete markets) capture the role of precautionary savings. Angeletos (2007) later showed that, even though aggregate assets are always higher under labor-income risk, they can be increased or decreased under capital-income risk (also called investment or production risk), even though $1 + r < 1/\beta$ still holds.

The relationship between the rate of return and the rate of discount also has implications for a long-standing debate on the return to equity and the return to safe assets. While the observed risk-free interest rates (i.e. those on Treasury bills), were around 1%, the average real return to equity was around 7%. This gap could not be matched by calibrating existing representative-agent models. In particular, these models predicted risk-free rates and equity premiums that were too large and too small, respectively. Huggett (1993) provides a partial answer for the relatively low risk-free rates by introducing uninsurable income risk. Because of the borrowing constraint, agents are restricted in the level of their indebtedness. However, agents are not restricted from accumulating saving. A low risk-free rate is then needed to persuade agents not to accumulate large credit balances so that the credit market can clear, as we will see in the next section. Angeletos (2007) revisits these results in a production economy with returns to private equity.

Closing the model: What about production?. To close the model we need to specify where output comes from. We will do this below looking at the two main alternatives: an endowment economy (without firms) as modeled in Huggett (1993) and a production economy as modeled in Aiyagari (1994). The firms are in any case kept as simple as possible, loading all the heterogeneity into the households.

10. An Endowment Economy

Consider an economy like the one described in Huggett (1993). The economy is populated by a continuum of households who receive a stochastic endowment of consumption goods (say, jelly beans) every period. Denote by ϵ the amount of jelly beans received by a household. The amount of jelly beans that a household receives follows a Markov process with transition function Q. The jelly beans follow independent but identical processes across households.

Sadly, jelly beans cannot be stored and thus have to be consumed the same period they are received. This is a problem for the households because they would much rather not have their income fluctuate every period. Nevertheless, the households can smooth consumption by contracting with one another, that is by exchanging jelly beans among them. This is, of course, an exchange economy (as opposed to the production economy we study in the next section).

The contracts that households can sign are limited in two ways. First, they cannot depend on the individual realizations of the present or future endowment of jelly beans. That is, the contracts can only specify a fixed amount of jelly beans being transferred today in exchange for a fixed amount in the future, regardless of how many jelly beans the household actually has. This form of market incompleteness can reflect information frictions, as the amount of jelly beans a household receives is their private information. Second, credit balances (receiving jelly beans today in exchange for payment of jelly beans in the future) are restricted, so that there is a limit to how many jelly beans can be obtained by a household in the market.

Therefore, the contracts take the form of saving/debt contracts, where the household exchanges an amount a' of jelly beans today in exchange for $(1+r)\,a'$ jelly beans in the future. When $a' > 0$ the household is saving and will receive payment in the future. When $a' < 0$ the household is in debt and receives jelly beans today in exchange for a payment in the future. This problem can be equivalently formulated in terms of (zero-coupon) bond contracts with a unit bond having a price of q. Letting $q = 1/1+r$ the problem would be buying a bond that pays one unit of future jelly beans paying q jelly beans today. This latter

formulation is the one used in Huggett (1993).

As explained above, it will be the case that households use financial contracts (bonds) to smooth consumption. They save if ϵ is high and borrow if ϵ is low.

The problem of the households is then

$$V(\epsilon, a) = \max_{\{c, a'\}} u(c) + \beta E\left[V\left(\epsilon', a'\right)|\epsilon\right]$$
$$\text{s.t. } c + a' = (1+r)a + \epsilon \qquad a' \geq \underline{a}$$

With respect to the workhorse model above, there is no labor choice and the "wage" is equal to 1, making income equal to ϵ. As before, households live forever and there is no aggregate risk. The Euler equation of a household is

$$u'(c) = \beta E_{\epsilon'}\left[V_a\left(\epsilon', a'\right)|\epsilon\right] = \beta(1+r)E\left[u'\left((1+r)a' + \epsilon' - g_a\left(\epsilon', a'\right)\right)\right],$$

where g_a is the policy function for savings.

The borrowing constraint must satisfy:

$$\underline{a} \geq -\epsilon_{min}/r.$$

This is called the *natural borrowing limit*. It is the highest amount of debt that can be rolled-over forever if the household faces an infinite sequence of the lowest possible endowment (or income shock).

Crucially, the financial contracts (bonds) are in zero net supply in this economy. This is an immediate consequence of the jelly beans not being storable. That is, there are no actual goods being transferred across time, only promises for future payments. The implication of this is that one household's savings are another household's debt. Even though the households trade against the market and not in bilateral meetings, the total amount of saving and debt in this economy most coincide. If this was a representative agent economy, there would be no trade!

Distribution of households. The fact that financial contracts are in zero net supply means that in equilibrium the aggregate level of assets (bonds) must be zero. To calculate this we need to define and obtain the distribution of households over states, Γ. Then we can calculate aggregate assets as the integral over the level of assets (bonds) with respect to the distribution of households (Γ). This is the key equilibrium condition for this economy.

Formally, we define $\overline{S}$ as the set of exogenous states and $[\underline{a}, \overline{a}]$ be the domain of assets

(Exercise: Show that there is an endogenous upper bound for the distribution of assets. See Huggett 1993, Theorem 2, Aiyagari 1994, footnote 18, Achdou et al. 2022, Proposition 2). Let $\mathcal{S}$ and $\mathcal{A}$ be σ-algebras over $\overline{S}$ and $[\underline{a}, \overline{a}]$ respectively. The distribution of agents is a measurable function $\Gamma : \mathcal{S} \times A \to [0, 1]$ that integrates to 1.

We can update the distribution of households by following their actions. Let $S \times A \in \mathcal{S} \times \mathcal{A}$ be a set in the σ-algebras. We want to know if there are households coming into the set $S \times A$. From the Markov kernel (or transition function) of ϵ we have the probability that $\epsilon' \in S$:

$$\Pr\left(\epsilon' \in S | \epsilon\right) = \int_{\epsilon' \in S} Q\left(\epsilon' | \epsilon\right) d\epsilon.$$

Then, we define an indicator function to know if $a' \in A$

$$g\left(\epsilon, a, A\right) = \begin{cases} 1 & \text{if } a' = g_a\left(\epsilon, a\right) \in A \\ 0 & \text{otw} \end{cases}.$$

With this we can update the distribution for all $(S, A) \in \mathcal{S} \times \mathcal{A}$:

$$\Gamma'\left(S, A\right) = \int_{\overline{S}} \int_{\overline{A}} \underbrace{g\left(\epsilon, a, A\right) \cdot \Pr\left(\epsilon' \in S | \epsilon\right)}_{\text{Markov Kernel:} P\left(\epsilon' a' | \epsilon, a\right)} \cdot d\Gamma\left(\epsilon, a\right).$$

This is the adjoint Markov operator of the Markov kernel (or transition function) Q of the joint process for (ϵ, a). The stationary distribution of states across households is the fixed point of this operator (Γ such that $\Gamma' = \Gamma$).

When the exogenous state is discrete (as is often the case) updating is simpler. We can work with: $\mathcal{S} = \{\{\epsilon_1\}, \ldots, \{\epsilon_n\}, \ldots, \{\epsilon_N\}\}$ instead of the σ-algebra and update according to

$$\Gamma'\left(\epsilon', A\right) = \sum_{\overline{S}} \int_{\overline{A}} g\left(\epsilon, a, A\right) \cdot \underbrace{Q\left(\epsilon' | \epsilon\right)}_{\text{Markov Transition Matrix}} \cdot d\Gamma\left(\epsilon, a\right)$$

Stationary recursive competitive equilibrium. A S-RCE is a set of a value function (V), a policy function $\left(a'\right)$, a distribution (Γ), and a price (r) such that:

(a) Given r the value and policy functions solve the agent's problem (the dynamic programming problem above).

(b) Given the policy function, Γ is a fixed point of the adjoint Markov operator of the endogenous Markov process defined above.

(c) Given the distribution and policy functions the asset (or bond) market clears:

$$\underbrace{\int\int a'(\epsilon, a) \cdot d\Gamma(\epsilon, a)}_{\text{Net Supply of Assets}} = 0 \longleftrightarrow \underbrace{\int\int c(\epsilon, a) \cdot d\Gamma(\epsilon, a)}_{\text{Demand for Goods}} = \underbrace{\int\int \epsilon \cdot d\Gamma(\epsilon, a)}_{\text{Supply of Goods}}$$

The definition of equilibrium also gives us an algorithm to find it.

input : Guess for price (r)
output : V, a', Γ, r

1. Solve the DP problem of the agent given r:
   $(V, a') = T(V; r)$ (a fixed point problem) ;
2. Find stationary distribution by iterating over the adjoint Markov operator ;
3. Check market clearing: $\sum_i \sum_j a'(\vec{\epsilon}_i, \vec{a}_j) \cdot \Gamma(i, j)$;
4. Update prices to clear market
   Manually by tatonnement or with a Root finder ;
5. Repeat (1)-(4) until market clears ;

Algorithm 4: S-RCE Algorithm

However, it is not a given that such an equilibrium exists. A couple of results are needed for this presented in Theorems 1 and 2 of Huggett (1993). If you read the explanation in the paper you will see how much of a group effort the development of these models was. The proofs of each result build on a series of concurrent papers. Interestingly, Huggett does not fully prove that an equilibrium exists. He conjectures that the excess supply of assets is increasing in the interest rate r (or decreasing in the price of bonds q) and uses tatonnment to search for an equilibrium. The conjecture is verified computationally but not proven.

11. A Production Economy

The economy in the previous section already showcases all the key ingredients of household heterogeneity. However, there is no production and hence no way to to talk about returns. The r in the previous model is better understood as an attribute of a financial contract, but not a rate of return on a productive investment. This ultimately prevents us from drawing a clear parallel between the heterogeneous agent economy above and the representative agent economy in the neoclassical growth model, where prices reflect the marginal productivity of factors used in generating the aggregate output of the economy.

The economy described in Aiyagari (1994) fills the next portion of the gap between heterogeneous agent models and the representative agent real business cycle models. The

key is the move from an endowment to a production economy. Production is (on purpose) just as in the neoclassical growth model: there is a representative firm operating a constant returns to scale technology, $Y = f(z, K, L)$. As it turns out, doing this greatly simplifies market clearing. The reason is in the discussion in Section 8.3. The scale of operations of a firm with a constant-returns-to-scale technology is undefined, and so prices have to be such that the firm is indifferent between any scale. For this to happen, prices must reflect the marginal product of inputs.

Aiyagari also interprets the value of ϵ as the value of "labor efficiency" instead of a direct income endowment. Under this interpretation, income depends on labor efficiency (ϵ) and the market wage (w). We assume for simplicity that individuals do not care about leisure and thus supply one unit of labor inelastically. As before, ϵ is stochastic and follows a Markov process Q. This is the only source of variation in the economy and the realizations of ϵ are independent across households.

The household's problem is then

$$V(\epsilon, a) = \max_{\{c, a'\}} u(c) + \beta E\left[V\left(\epsilon', a'\right)|\epsilon\right]$$
$$\text{s.t. } c + a' = (1+r)a + w\epsilon \qquad a' \geq \underline{a}$$

The borrowing limit must satisfy

$$\overline{a} \geq -\frac{w\epsilon_{min}}{r}.$$

The Euler equation is, as before,

$$u'((1+r)a + w\epsilon) = \beta E\left[V_a\left(\epsilon', a'\right)|\epsilon\right].$$

As above, the households take the (constant) prices as given. Their behavior induces a Markov process over the state (ϵ, a) that is constructed in the same way as in the previous section and results in a stationary distribution of households Γ.

The firm's cost minimization problem implies that the demand for inputs is perfectly elastic at equilibrium prices. Therefore, the only way to clear the market is to make producers indifferent between any scale of production by setting

$$r = f_k(z, K, L) - \delta \qquad w = f_L(z, K, L).$$

Stationary recursive competitive equilibrium. A S-RCE is a set of a value function (V), a policy function $\left(a'\right)$, a distribution (Γ), and prices (r, w) such that:

(a) Given (r, w) the value and policy functions solve the agent's problem (the dynamic programming problem above).

(b) Given the policy function, Γ is a fixed point of the adjoint Markov operator of the endogenous Markov process defined by the (exogenous) Markov process for ϵ with Transition function Q and the (endogenous) policy function for a'.

(c) The capital and labor markets clear. That is, aggregate capital reflects the aggregate assets of the households and aggregate labor demand equals the supply of "efficiency units of labor" from the household

$$K = \int \int a \cdot d\Gamma(\epsilon, a) \qquad L = \int \int \epsilon \cdot d\Gamma(\epsilon, a).$$

(d) Prices are consistent with firm optimization (in this case cost-minimization)

$$r = f_k(z, K, L) - \delta \qquad w = f_L(z, K, L).$$

input : Guess for price (r)
output : V, a', Γ, r

1. Solve the DP problem of the agent given (r, w):
   $(V, a') = T(V; r)$ (a fixed point problem) ;
2. Find stationary distribution method (or update dist. N times) ;
3. Update prices to ensure market clearing:
   $K = \sum_i \sum_j a \cdot \Gamma(i, j) \quad \longrightarrow \quad r = f_K(z, K, 1) - \delta \quad w = f_L(z, K, 1)$;
   Dampen updating of prices if necessary ;
4. Repeat (1)-(3) until prices converge ;

Algorithm 5: S-RCE Algorithm

Note that the labor supply is exogenous! So, (i) we can simplify the integral because it does not depend on the level of assets, and (ii) we can normalize the level of ϵ so that the market clearing condition becomes: $L = \int \epsilon \Gamma_\epsilon(\epsilon)\, d\epsilon = 1$, where Γ_ϵ is the stationary distribution of the Markov process Q having normalized the levels so that the expected value of ϵ is 1.

Finally, the definition of equilibrium also gives us an algorithm to find it. The key is that once we normalize the level of labor supply we realize that, just as in the neoclassical growth model, all the economy depends only on the aggregate level of capital, K. Once K is known it implies prices $r = f_k(z, K, 1) - \delta$ and $w = f_L(z, K, 1)$. With this prices we can

solve the household's problem and then arrive at at new level of K. This means that the problem of finding an equilibrium is a fixed point problem on a single value, the value of aggregate capital. Incidentally, a good initial condition is the steady state value of capital in the corresponding complete-markets/representative-agent economy.

12. A Production Economy with Aggregate Risk

There is, of course one critical aspect missing from the previous models: aggregate risk. In all of them, there is only idiosyncratic variation affecting households, but leaving aggregates unchanged. Hence the focus on the stationary recursive competitive equilibria of those economies.

The next step is to introduce aggregate fluctuations into the Aiyagari economy. Krusell and Smith (1998) is the first paper to accomplish this taking (essentially) the same economy presented in the previous question and making z, the productivity of the firm, stochastic. Assume it follows a Markov process with transition function Q_z. We will write the (recursive) household problem later, but first we need to discuss the implications of aggregate shocks.

The re-introduction of aggregate productivity means that we are back at the problem of Section 8, but with the added difficulty of having a time-varying distribution of households. The key issue is, as before, determining what is the relevant state of the household problem. The state must (*i*) contain all the information needed to solve the problem, including prices, and (*ii*) be updated to compute continuation values, as well as expectations over prices and other variables.

In Section 8 this problem was solved by keeping track of the aggregate capital stock K and imposing consistency between the transition function of the aggregate capital stock and the savings function of the representative household ($G_K = g_a$). The key was that the aggregate capital stock was the only object needed to compute prices and solve the household problem.

Unfortunately, there is no longer an immediate correspondence between aggregate capital and individual assets. Aggregate capital is still enough to compute prices (as discussed at the end of the previous section), but to know its value we have no option but to integrate over the asset holdings of households with respect to the (time-varying) distribution of households

$$K_t = \int \int a \cdot d\Gamma_t (\epsilon, a) .$$

Further, in order to update the aggregate capital, it becomes necessary to update the distribution itself $\Gamma_{t+1} = T_t^\star \Gamma_t$ using the adjoint Markov operator of the process induced by households' saving choices.

The outcome of this is that aggregate capital is no longer a viable state and that instead the households need to keep track of the whole distribution Γ and its evolution. However, the evolution of the distribution itself is state dependent as it reflects the households' saving choices that respond to the aggregate state of the economy.

Recursive competitive equilibrium. A RCE is a set of a value function (V), a policy function $\left(a^{'}\right)$, a state-dependent transition function for the distribution ($T^{\star}$), and price functions (R, W) such that:

(a) Given the functions (R, W) and the transition function for the distribution ($T^{\star}$), the value and policy functions solve the households' dynamic programming problem

$$V\left(\underbrace{\epsilon, a}_{\text{Ind. State}}, \underbrace{z, \Gamma}_{\text{Agg. State}}\right) = \max_{\left\{c, a^{'}\right\}} u(c) + \beta E\left[V\left(\epsilon^{'}, a^{'}, z^{'}, \Gamma^{'}\right) | \epsilon, z, \Gamma\right]$$
$$\text{s.t. } c + a^{'} = (1 + R(z, \Gamma))\, a + W(z, \Gamma)\, \epsilon \qquad a^{'} \geq \underline{a} \qquad \Gamma^{'} = T^{\star}\Gamma$$

$T^{\star}$'s dependence on the aggregate state is omitted from the notation for simplicity. Alternatively, we can write $\Gamma^{'} = H(z, \Gamma)$ for a transition function H. We can also write prices as functions of productivity and aggregate capital, $r = R(z, K)$ and $w = W(z, K)$.

(b) The transition function for the distribution of agents corresponds to the adjoint Markov operator of the endogenous Markov process defined by the (exogenous) Markov processes for ϵ and z with Transition function Q and Q_z and the (endogenous) policy function for $a^{'}$.

$$\Gamma^{'}(S_\epsilon, A) = T^{\star}\Gamma(S_\epsilon, A) = \int_{\overline{S}} \int_{\overline{A}} \underbrace{g(\epsilon, a, z, \Gamma) \cdot \Pr\left(\epsilon^{'} \in S_\epsilon | \epsilon\right)}_{\text{Markov Kernel: } P\left(\epsilon^{'}, a^{'} | \epsilon, a; z, \Gamma\right)} \cdot d\Gamma(\epsilon, a)$$

where

$$g(\epsilon, a, z, \Gamma, A) = \begin{cases} 1 & \text{if } a^{'} = g_a(\epsilon, a, z, \Gamma) \in A \\ 0 & \text{otw} \end{cases}.$$

(c) The capital and labor markets clear. That is, aggregate capital reflects the aggregate assets of the households and aggregate labor demand equals the supply of "efficiency units of labor" from the household

$$K = \int \int a \cdot d\Gamma(\epsilon, a) \qquad L = \int \int \epsilon \cdot d\Gamma(\epsilon, a) \, (= 1).$$

Note that the aggregate capital is a function of the distribution of households, $K(\Gamma)$.

(d) Prices are consistent with firm optimization (in this case cost-minimization)

$$r = f_k(z, K, L) - \delta \qquad w = f_L(z, K, L).$$

This implies

$$R(z, K) = f_k(z, K, 1) - \delta \qquad W(z, K) = f_L(z, K, 1).$$

The version with prices depending on the distribution Γ rather than on aggregate capital makes it explicit that the underlying state determining capital is the distribution Γ.

The definition of the RCE makes the problems introduced by aggregate fluctuations apparent. It is impossible to keep track of the whole distribution of households and generate a state-dependent transition function that is consistent with household optimization.

The main break-through in Krusell and Smith (1998) is providing a computational method that can approximate the solution to the RCE. The key of the algorithm is already present in the discussion of the RCE in Section 8 and of the S-RCE of the Aiyagari economy above. The distribution of households is only really needed to compute the aggregate capital level, K. Notably, this corresponds to the first moment of the wealth distribution. The reason only aggregate capital is needed is that it is sufficient to compute prices. The challenge is that we also need to know how aggregate capital evolves, that is, we need a transition function for aggregate capital. However, the evolution of capital depends on the whole distribution.

Krusell and Smith (1998) solve the problem by proposing an approximate law of motion for capital that depends on moments of the wealth distribution. In principle, if enough moments are taken into account they can sufficiently summarize the information in the wealth distribution. If the approximation works the problem of keeping track of the entire distribution can be reduced to the (much simpler) problem of keeping track of a finite set of moments.

In practice, the rational expectation assumption is partly lifted. Instead of forming exact expectations over the distribution of wealth with knowledge of the transition function $T^\star$, households use a (log-)linear model to forecast future moments of the distribution using its current moments and the aggregate productivity. For instance

$$\log K' = f\left(z, \log K, (\log K)^2\right) = \alpha(z) + \beta(z) \log K + \gamma(z) (\log K)^2,$$

where the coefficients depend on the values of z. The solution method proposed in the paper

finds coefficients that (almost exactly) match the evolution of capital. When z is discrete there are finitely many coefficients to find. The surprising result in the implementation of the solution is that $\gamma(z) = 0$, implying that only the first moment is required to approximate the evolution of aggregate capital.

input : Guess for coeficients $\alpha(z)$, $\beta(z)$
output : V, a', Γ

1. Solve the DP problem of the agent given market clearing prices $(R(z, K), W(z, K))$ and the transition function for K parameterized by $\alpha(z)$, $\beta(z)$:
   $(V, a') = T(V; \alpha, \beta)$ (a fixed point problem). Note that markets will clear by construction. ;
2. Use the policy function and the Markov processes for ϵ and z to simulate a long panel of the economy. In doing this update the distribution using the appropriate Markov operator. Use the distribution to compute aggregate capital. Record the time series for aggregate capital and productivity. ;
3. Regress capital on moments to update the coefficients allowing for state-dependent coefficients by the level of productivity. ;
4. Repeat (1)-(3) until prices converge and guarantee that the R^2 of the regression is close to 1. ;

Algorithm 6: Krusell-Smith Algorithm

Approximate aggregation. The main result in Krusell and Smith (1998) is that, as it turns out, it is sufficient to only keep track of the first moment of the wealth distribution in order to accurately approximate the aggregate states of the model. That is, aggregate capital is (approximately) the relevant aggregate state for the households in the economy, just at in the neoclassical growth model of Section 8. However, we know while there is exact aggregation in the neoclassical growth model (allowing for a representative agent), there is no such aggregation result in the Aiyagari economy. This led Krusell and Smith to term their result "*approximate aggregation*." They write

> "Our main insight is that the macroeconomic model with heterogeneity features approximate aggregation. By approximate aggregation, we mean that, in equilibrium, all aggregate variables—consumption, the capital stock, and relative prices—can be almost perfectly de- scribed as a function of two simple statistics: the mean of the wealth distribution and the aggregate productivity shock."

This result makes the model solvable in practice by reducing the dimensionality of the problem.

Where does approximate aggregation come from? Why does it work? The key is that even with a single non-state-contingent asset the households can achieve a great deal of insurance, effectively smoothing out the fluctuations in their marginal utility of consumption, in the same way that a representative household would. The outcome of this is that households in the heterogeneous agent economy behave (approximately) as scaled-versions of the representative household (except for those that are borrowing-constrained). Krusell and Smith (1998) explain it as follows:

> "The key insight is related to earlier findings from similar models that utility costs from fluctuations in consumption are quite small and **that self-insurance with only one asset is quite effective**. Self-insurance in our model is not very effective in terms of smoothing individual relative to aggregate consumption; for example, the unconditional standard deviation of individual consumption is about four times that of aggregate consumption, and the unconditional correlation of the consumption of any two agents is very close to zero. However, **in utility terms, agents in our stationary equilibria are insured well enough that the marginal propensity to save out of current wealth is almost completely independent of the levels of wealth and labor income**, except at the very lowest levels of wealth. Furthermore, although some very poor agents have substantially different marginal savings propensities at any point in time, the fraction of total wealth held by these agents is always very small (this is particularly true in the model with a realistic wealth distribution). Because it is so small, **higher-order moments of the wealth distribution simply do not affect the accumulation pattern of total capital, even though these moments do move significantly over time**." [emphasis added]

This is (at its core) the same reason behind the inability of Aiyagari economies based on labor-income risk to generate realistic distributions of wealth, as explained in Benhabib and Bisin (2018) and Stachurski and Akira Toda (2019, 2020). As households' move away from the collateral constraint, their saving functions tend to be linear. This limit behavior means that households can be (approximately) aggregated and is what explains the similar marginal propensity to consume referenced in the passage above.

The approximate aggregation result has also misleadingly led to thinking that heterogeneity does not matter for aggregate fluctuations. That is not the case as shown in Krusell and Smith (1998) and their concurrent paper Krusell and Smith (1997). Approximate aggregation follows from the limit behavior of saving rates in the model, and not from the irrelevance of heterogeneity. Rather, it is because the baseline Aiyagari

economy inability to reproduce the observed levels of wealth inequality that Krusell and Smith obtain their (in)famous irrelevance result. They write

> "When the representative-agent model is altered only by adding idiosyncratic, uninsurable risk, the resulting stationary wealth distribution is quite unrealistic: there are too few very poor agents, and much too little concentration of wealth among the very richest. For this reason, we consider a version of the model with preference heterogeneity [...] We show that this model does succeed quite well in matching the key features of the wealth distribution. [...] in the aggregate, we observe a significant departure from permanent income behavior, in contrast to standard representative-agent models."

In fact, the extended model in Krusell and Smith (1997) also attempts to better match the observed distribution of wealth (an elusive target for the literature). They also find a relevant role for heterogeneity in shaping the aggregates of the economy. The reason wealth concentration matters is that it generates (endogenously) a mass of households that act in a "hand-to-mouth" fashion. These households are not able to effectively insure against aggregate fluctuations and behave markedly different from the representative agent (that is never against the borrowing constraint and behaves like a permanent-income agent, smoothing consumption).

Similarly, Angeletos (2007) shows that introducing capital-income risk in the form of risky portfolios implies significant departures from the representative agent baseline. He writes

> "[S]ignificant general-equilibrium effects on savings and income are both empirically plausible and consistent with low private-equity premia [...] the macroeconomic effects of idiosyncratic investment risk can be both qualitatively distinct from those of idiosyncratic labor-income risk and quantitatively significant."

Other papers that highlight the importance of heterogeneity for aggregate fluctuations are: Kaplan, Moll, and Violante (2018), Ahn, Kaplan, Moll, Winberry, and Wolf (2018) Kaplan and Violante (2022). See Kaplan and Violante (2018) in the Journal of Economic Perspectives for a non-technical summary of the HANK literature. One important aspect of these papers is highlight the role of portfolio composition and illiquid assets in generating a group of wealthy-hand-to-mouth households who, despite being at the top of the wealth distribution, behave as borrowing constraint agents because their wealth is not readily accessible to produce insurance (as is the case with housing).

Modern solution methods. In a concurrent article, Den Haan (1997) develops an alternative solution method that parametrizes the distribution of wealth on a standard polynomial basis, this reduces the dimensionality of the problem. Rather than track the infinite-dimensional distribution and its transition function, Den Haan's method searches for vector of parameters that approximate the functions on a given polynomial basis. One important advantage of this method is that it does not rely on imprecise Monte Carlo simulations of the model in order to obtain a solution. This method has been extended, for instance in Winberry (2018).

An alternative that has proven to be effective is using Perturbation methods, as in Reiter (2009). The best application of this is Auclert, Bardoczy, Rognlie, and Straub (2021) which is currently the best method to solve heterogeneous agent models with aggregate fluctuations.

Part IV

Firm Heterogeneity

We now focus on the firm problem and one of two main ways in which we can introduce firm heterogeneity into macroeconomic models. Technically, the impediment to meaningful firm heterogeneity lies in the combination of constant returns to scale and price-taking (or perfect competition) behavior. These features imply that the choices of all firms are scaled-versions of one-another, allowing for exact aggregation of firms into a representative firm. Departing from either of these features generates curvature in the firm's problem, making firms of different sizes behave differently. We go over the basics of each approach.

13. Span of Control

Models following the work of Lucas (1978a) and Hopenhayn (1992) have competitive firms that operate technologies with decreasing returns to scale and whose owners (or managers) differ in their "managerial talent." This is captured by differences in firms' productivity (z_i). More "talented" managers have a larger "span of control," that is, their optimal firm size is larger. The decreasing returns to scale determine how differences in talent (productivity) translate into differences in size, but they also imply that firms make profits in equilibrium. However, these profits are not the result of market power but of the returns to the "managerial input." These are rents to the fixed input that the manager owns.

While we will not cover these models in detail, we a brief description of the problem is useful.[10] I focus on Lucas (1978a). Lucas' model is static and is concerned with explaining the distribution of firms. Consider a firm that produces in two stages, first it combines capital and labor with a standard constant-returns-to-scale technology, $y = F(k, n)$, then that output is mixed with managerial talent to produce output, so that total output is

$$z_i g\left(y_i\right) = z_i g\left(F\left(k_i, n_i\right)\right)$$

where g is strictly increasing and strictly concave. The managerial talent z_i is the fixed input and the concavity of g implies that there are decreasing returns to scale to the firm. The practical implication of the decreasing returns is that (unlike with constant returns to scale) the best manager (highest z) cannot use all the resources.

[10] This description follows the lecture notes by Chris Edmond: http://www.chrisedmond.net/phd2014/90065_lecture1.pdf.

We can simplify the problem by using the homogeneity of F to write output as

$$z_i g\left(y_i\right) = z_i g\left(f\left(\kappa_i\right) n_i\right),$$

where $\kappa = k/n$ is capital per-worker and $f(x) = F(x, 1)$. This turns to be convenient because we isolate the labor demand of firms as being affected directly by g. This matters because the objective of this theory is to explain the distribution of firm size, with size measured by employment.

The problem of a firm manager is then to choose labor and capital-per-worker:

$$\pi(z) = \max_{\kappa, n} zg(f(\kappa) n) - wn - r\kappa n$$

(note that without g this problem would be linear in n, leading to corner solution in scale)

The first order conditions are

$$zg^{'}(f(\kappa) n) f^{'}(\kappa) = r$$
$$zg^{'}(f(\kappa) n) f(\kappa) = w + r\kappa$$

We can use these conditions to solve for the optimal capital-per-worker. Dividing them gets us

$$\frac{f(\kappa)}{f^{'}(\kappa)} - \kappa = \frac{w}{r}$$

Recall that firms face competitive input markets, so prices w and r are common across firms. They also operate the same technology. The only difference is in managerial talent (z). So, this equations implicitly defines the optimal κ for all firms. Lets call that level $\kappa^{\star}$.

The scale of the firm is then pinned down by solving for $y^{\star}(z)$ out of

$$zg^{'}\left(y^{\star}(z)\right) f^{'}\left(\kappa^{\star}\right) = r$$

once we have $y^{\star}(z)$ we can also get $n^{\star}(z) = y^{\star}(z)/f(\kappa^{\star})$.

Lucas then uses this solution to ask about the properties of g that are needed for the distribution of firms to satisfy the patterns observed in the data. The most salient feature is known as Gibrat's law and it captures the fact that firm growth is independent of firm size. In particular, as factor prices change, firms (employment) will grow (or shrink). This growth should be independent of their (employment) size.

The size of the firm is captured by $n^{\star}\left(z_i, w, r\right)$. Different firms have different z and hence different sizes, but they all face the same prices. The total differential of the function

$\ln n^\star$ as factor prices change gives us the growth rate of firm size,

$$d \ln n^\star (z, w, r) = \frac{n^\star_w (z, w, r)}{n^\star (z, w, r)} dw + \frac{n^\star_r (z, w, r)}{n^\star (z, w, r)} dr.$$

For this to be independent of size it must be that

$$\frac{\partial \left(d \ln n^\star (z, w, r) \right)}{\partial z} = 0$$

for that to always be the case it must be that

$$\frac{\partial}{\partial z} \left(\frac{n^\star_w (z, w, r)}{n^\star (z, w, r)} \right) = 0 \qquad \text{and} \qquad \frac{\partial}{\partial z} \left(\frac{n^\star_r (z, w, r)}{n^\star (z, w, r)} \right) = 0$$

These conditions impose restrictions over the shape of g. Lucas solves these differential equations and finds that the only function that satisfies them is $g(x) = Ax^\alpha$ for some constant $A > 0$ and a power $\alpha \in (0, 1)$. We can verify this directly. The optimal (inner) output size is

$$y^\star (z) = \left(\alpha A z \frac{f' (\kappa^\star)}{r} \right)^{\frac{1}{1-\alpha}}$$

and so the optimal labor demand is

$$n^\star (z) = \left(\frac{\alpha A z}{r} \right)^{\frac{1}{1-\alpha}} \left(f' \left(\kappa^\star \right) \right)^{\frac{\alpha}{1-\alpha}} \left(\frac{w}{r} + \kappa^\star \right).$$

Even though this expression looks daunting, it delivers the result we want immediately, because it implies that the labor (size) of the firm is log-separable in z and the terms that depend on prices $(w, r, \kappa^\star)$.

$$\ln n^\star (z) = \frac{1}{1 - \alpha} \ln z + \ln h (r, w)$$

which deliver s the result required by Gibrat's law.

So we have a single parameter α that determines the (decreasing) returns to scale of the firms. The scale parameter A plays no real role in these results and so we normalize to 1.

We can further model firm entry as a reflection of the managers' occupational choice. People have different managerial talent, but they all have the same skills as a worker. If a person can supply their (full) time as either a worker or a manager then they will choose whichever delivers them the highest income (absent any preferences or amenities, as is the case in this model). The income as a worker is given by the wage, w, and is constant across

people. The income as a manager is given by the managerial profits, $\pi(z)$, and depends on talent. There is then a cutoff for the managerial talent, above which people become managers. That is, a $z^\star$ such that

$$\begin{aligned} \pi\left(z^\star\right) &= w \\ \underbrace{z^\star g\left(y^\star\right)}_{\text{Revenue}} &= \underbrace{w}_{\text{Fixed Cost}} + \underbrace{wn^\star - r\kappa^\star n^\star}_{\text{Variable Costs}} \end{aligned}$$

This is a zero profit condition for the marginal manager that takes into account their opportunity cost of not being a worker. That constitutes the fixed cost of setting up a firm.

The work of Hopenhayn (1992) and the literature that it generated extends these ideas to a dynamic setting. It also takes into account the role of fixed set-up costs in determining firm entry.

A special case. We can further simply the problem if we assume that the firm operates only with labor, so that $y = n$. Then the managerial technology is $z_i n^\alpha$. Because there is no capital, the first order condition is now $zg'(n) = w$. Solving it gives us a closed-form expression for the size of the firm as a function of managerial talent:

$$n^\star(z) = \left(\frac{\alpha z}{w}\right)^{\frac{1}{1-\alpha}}.$$

If we further assume that managerial talent is distributed Pareto with a CDF $1 - z^{-\xi}$, the distribution of firm size is also Pareto, but with a Pareto parameter $\xi(1-\alpha)$.

To see this recall that the defining feature of the Pareto distribution is that its counter-CDF has the form Bn^{-p}, with p the Pareto parameter and B a constant. so we want to know p from

$$n^{-p} = 1 - \text{CDF}(n) = \Pr(\tilde{n} > n) = \Pr\left(\tilde{z} > \frac{w}{\alpha} n^{1-\alpha}\right)$$

the last equality says that for a firm to have more than n employees ($\tilde{n} > n$) it must have managerial talent $\tilde{z}$ above $\frac{w}{\alpha} n^{1-\alpha}$. We know the share of firms satisfying that condition:

$$\underbrace{\Pr\left(\tilde{z} > \frac{w}{\alpha} n^{1-\alpha}\right)}_{\text{Counter CDF of } z} = \left(\frac{w}{\alpha} n^{1-\alpha}\right)^{-\xi} = \left(\frac{w}{\alpha}\right)^{-\xi} n^{-\xi(1-\alpha)}$$

this gives us our result. The employment of firms is distributed Pareto with power $p = \xi(1-\alpha)$.

This result gives us insight over the role of the market (in this case through the returns

to managerial talent) in amplifying differences across firms (although similar insights apply to differences in income or wealth across individuals). First, note that all profits in this environment are the return to the fixed (managerial) factor. The technology $z_i n^\alpha$ makes these returns to be $1-\alpha$ of total output (just as in the general Cobb-Douglas case, just define $x_i = z_i^{\frac{1}{1-\alpha}}$ to write output as $x^{1-\alpha}n^\alpha$). Then, note that the returns to scale amplify differences in scale as $\alpha \to 1$. That is, the more scalable the technology, the larger the differences between firms:

$$\underbrace{\ln n_i - \ln n_j}_{\text{Diff. in Scale/Size}} = \underbrace{\frac{1}{1-\alpha}}_{\text{Tech. Amplification}} \times \underbrace{\left(\ln z_i - \ln z_j\right)}_{\text{Diff. in Managerial Talent}} .$$

We should therefore expect larger differences in size (thicker right tails) in industries with more scalable technologies (higher α).

14. Market Power

An alternative approach to introduce curvature into the firms' problem is to acknowledge the presence of market power in the form of price-setting behavior. The choice over prices introduces curvature over the firms' revenue. Under perfect competition revenue is linear in quantities as price per unit is constant (linear pricing). Market power introduces curvature because it takes into account the fact that the demand for goods is decreasing in prices, and that it is therefore necessary to reduce prices to increase quantities. The curvature is introduced into the revenue function of the firm, rather than in production. This in turn generates differences across firms as firms with higher productivity can charge lower prices and operate at a larger scale.

This approach also results in firms charging markups and making profits. This has made it widely used in many areas such as the New-Keynesian literature, international trade, and growth, where price-setting and the profit-motive play a central role.

Markups and firm heterogeneity. We abandon the price-taking assumption we have imposed so far on firms and instead study firms that choose prices as well as inputs.

The firm's problem is to maximize its profits taking as given an (inverse) demand curve. There are two options for the type of competition faced by the firm: Bertrand competition where firms choose prices (p) and Cournot competition where firms choose over quantities (y). In either case the firms use the demand curve they face to map the choice over one variable into the effects on the other (say, what is the quantity demand if a given price is

chosen). We introduce a sub-index i to denote the firm.

$$\max \; p_i y_i - C_i\left(y_i\right) \qquad \text{s.t. } p_i = D_i\left(y_i\right)$$

where $C_i(y)$ gives the cost to firm i of producing y units of the good and $D_i(y)$ is the (inverse) demand faced by firm i. We show a special case of the cost function below and explore various demand systems that shape markups in the sub-sections that follow.

The optimal behavior of the firm does not depend on whether it is choosing prices à la Bertrand or quantities à la Cournot. The optimal choice is to set prices as a markup over the marginal cost, with the markup determined by the elasticity of demand $\varepsilon_i \equiv -\left(\frac{\partial \log p_i}{\partial \log y_i}\right)^{-1}$,

$$p_i = \frac{1}{1-\frac{1}{\varepsilon_i}} C_i'\left(y_i\right),$$

The markup is $\mu_i^m = 1/1-\frac{1}{\varepsilon_i}$. The difference between forms of competition is the value of the elasticity of demand and hence markups, prices, and output levels.

The implication of this result is that the scale of production can differ across firm for two reasons: they can have different marginal costs (as captured by differences in their cost function C_i) or face different elasticities of demand (as captured by ε_i). Of course both reasons can be active at the same time. Firms can differ in their productivity, leading to differences in scale that can affect the elasticity of demand. The elasticity is in general not constant and can depend on how much of a good is being demanded. There can also be differences in preferences for goods that make the elasticity of one firm higher or lower than that of other goods.

The models shown in the following sections incorporate these ideas into markets where there is competition among producers of different varieties of goods.

Equivalence of Bertrand and Cournot competition. Lets start with Bertrand. The first order condition of the firm is

$$\begin{aligned}
\frac{\partial}{\partial p_i}\left(p_i y_i - C_i\left(y_i\right)\right) &= 0 \\
y_i + p_i \frac{\partial y_i}{\partial p_i} - C_i'\left(y_i\right)\frac{\partial y_i}{\partial p_i} &= 0 \\
1 + \frac{p_i}{y_i}\frac{\partial y_i}{\partial p_i} - \frac{C_i'\left(y_i\right)}{p_i}\frac{p_i}{y_i}\frac{\partial y_i}{\partial p_i} &= 0
\end{aligned}$$

$$
\begin{aligned}
1 + \left(\frac{\partial \log p_i}{\partial \log y_i} \right)^{-1} \left(1 - \frac{C_i'(y_i)}{p_i} \right) &= 0 \\
1 - \varepsilon_i \left(1 - \frac{C_i'(y_i)}{p_i} \right) &= 0 \\
\frac{1}{1 - \frac{1}{\varepsilon_i}} C_i'(y_i) &= p_i
\end{aligned}
$$

The same result arises under Cournot (but in fewer steps).

$$
\begin{aligned}
\frac{\partial}{\partial y_i} \left(p_i y_i - C_i(y_i) \right) &= 0 \\
p_i + y_i \frac{\partial p_i}{\partial y_i} - C_i'(y_i) &= 0 \\
1 + \frac{y_i}{p_i} \frac{\partial p_i}{\partial y_i} - \frac{C_i'(y_i)}{p_i} &= 0 \\
1 - \frac{1}{\varepsilon_i} - \frac{C_i'(y_i)}{p_i} &= 0 \\
\frac{1}{1 - \frac{1}{\varepsilon_i}} C_i'(y_i) &= p_i
\end{aligned}
$$

Cost minimization with constant-returns-to-scale. It is useful to separate the problem of the firm into an inner-step of cost minimization that produces the cost function and an outer-step of profit maximization that gives markups (as above).

A special case arises when the production function of the firm has constant returns to scale. In this case the marginal cost is constant, that is, $C_i'(y) = c_i$. The linearity of the cost function is an immediate consequence of the cost function inheriting the homogeneity of degree one of the production function.

Moreover, we can choose a single input, say labor, and express all choices of the firm directly over that input, as the value of other inputs and of total output vary linearly with it. This is the underlying reason for many applications to focus on simple production functions that are linear on labor and have a marginal cost of w/z, with z being the firm's productivity.

Consider the problem of a firm that use multiple inputs $\{x_k\}_{k=1}^K$ in addition to labor to produce:

$$
y_i = \tilde{z}_i F\left(x_1, \ldots, x_K, n_i\right),
$$

where the function F is strictly concave, twice continuously differentiable, and has constant

returns to scale.

Let the prices of inputs be $\left\{\tilde{p}_k\right\}_{k=1}^{K}$ and $\tilde{w}_\ell$ respectively. We know from the firm's optimality condition that:

$$\tilde{z}_i F_k\left(\frac{x_1}{n_i}, \ldots, \frac{x_K}{n_i}, 1\right) = \tilde{p}_k$$

recalling that, because of Euler's theorem, F_k is homogeneous of degree zero for every k. These equations define a square system in the ratio ratio of each input x_k to labor. The system has a solution that gives the ratios in terms of parameters:

$$\frac{x_k}{n_i^{j\ell}} = g_k\left(\tilde{z}_i^{j\ell}, \tilde{p}_1, \ldots, \tilde{p}_K\right)$$

The existence of a solution follows from the inverse function theorem applied to the function $\nabla_x F : \mathbb{R}_{++}^K \to \mathbb{R}_{++}^K$, where the operator ∇_x gives the first derivatives of F with respect to the variables $\left\{x_k\right\}_{k=1}^{K}$. Note that the Jacobian of ∇F is given by the first K rows and columns of the Hessian of F, which is negative definite for all interior points by the strict concavity of F. The negative definiteness of the Jacobian ensures the invertibility of $\nabla_x F$.

Given the system's solution we express the production function in terms of labor alone:

$$y_i = \tilde{z}_i F\left(\frac{x_1}{n_i}, \ldots, \frac{x_K}{n_i}, 1\right) n_i = z_i n_i$$

where we define the effective productivity of labor as $z_i \equiv \tilde{z}_i F\left(x_1/n_i, \ldots, x_K/n_i, 1\right)$ with the ratios x_k/n_i given by the solution to the system defined above. Thus, z_i is a function of productivity $\tilde{z}_i$ and the price of the other inputs. It is now possible to use this linear production function in labor instead of the original production function F on many inputs.

Finally, the cost of labor must take into account that other inputs react to changes in labor according to the proportionality defined by $\left\{g_k\right\}_{k=1}^{K}$. Then, the cost of the firm is given by:

$$\sum_{k=1}^{K} \tilde{p}_k x_k + \tilde{w} n_i = \underbrace{\left(\sum_{k=1}^{K} \tilde{p}_k \frac{x_k}{n_i} + \tilde{w}\right)}_{\text{Effective Labor Cost } w} n_i = w n_i$$

where wn_i represents the cost of goods sold, and w is not directly the wage, but a measure of costs that takes into account the price of other inputs and the change in their demand in response to changes in the firm's labor demand.

Finally, it is equivalent to express the problem in terms of labor or output as profits can be written as

$$p_i y_i - \underbrace{\frac{w}{z_i} y_i}_{C_i(y_i)} \longleftrightarrow p_i z_i n_i - w n_i$$

Accordingly, we will simplify the notation in the rest of this section and work with a linear cost function $C_i(y) = c_i y$.

15. Monopolistic Competition

We now look at what is by far the most popular demand system for applied macroeconomics: Monopolistic Competition. The model as used now was developed in Dixit and Stiglitz (1977) and captures a large number of firms producing differentiated goods that are nonetheless (imperfect) substitutes for one another, following insights from Chamberlain (1933). The fact that the goods are differentiated means that individual producers haver market power over their variety, but the substitutability with other products places limits on this power as producers compete for higher demand for their individual variety of good. In the original formulation there was a finite (but large) number of producers. We will instead use the more common formulation of a continuum of goods indexed by $i \in [0, 1]$.

At the center of the model are the preferences for the various products, or equivalently, an aggregation technology that combines individual varieties to produce a single final good. In the first case, demand comes from identical consumers that maximize the utility they derive from the consumption of the different good varieties

$$U = \left(\int_0^1 y_i^{\frac{\varepsilon-1}{\varepsilon}} di \right)^{\frac{\varepsilon}{\varepsilon-1}}.$$

In the second case the integral is equal to the value of the aggregate good Y. This second version is more common in applications in international trade and New-Keynesian models where the final good is demanded along with foreign goods or where utility also includes leisure.

The parameter $\varepsilon > 1$ plays an crucial role as it determines the degree of substitutability between goods and, as we will see later, the elasticity of demand.

The problem of the consumer (or final good producer) is to minimize the cost of their consumption basket (or inputs) subject to some desired utility (or output) level. Notice that the utility (or production) function has constant returns to scale and thus the maximization

problem is ill-defined:

$$\min_{\{y_i\}} \int_0^1 p_i y_i di - \lambda \left(\left(\int_0^1 y_i^{\frac{\varepsilon-1}{\varepsilon}} di \right)^{\frac{\varepsilon}{\varepsilon-1}} - \overline{Y} \right),$$

with $\lambda > 0$ a Lagrange multiplier and $\overline{Y}$ is the desired level of utility.

The first order conditions are, for any i,

$$p_i = \lambda \left(\int_0^1 y_i^{\frac{\varepsilon-1}{\varepsilon}} di \right)^{\frac{\varepsilon}{\varepsilon-1}-1} y_i^{\frac{-1}{\varepsilon}}.$$

Hence, optimality requires that between any pair of goods (i, j)

$$\underbrace{\frac{p_i}{p_j}}_{\text{MRT}} = \underbrace{\left(\frac{y_i}{y_j} \right)^{\frac{-1}{\varepsilon}}}_{\text{MRS}} \longrightarrow \frac{y_i}{y_j} = \left(\frac{p_i}{p_j} \right)^{-\varepsilon}.$$

From this we see that $\varepsilon = \frac{\log y_i}{\log p_j}$ is the elasticity of substitution between goods i and j.[11] Moreover, the elasticity of substitution is constant across pairs. This is what gives its name to the aggregator as the Constant-Elasticity-of-Substitution (CES) aggregator.

Going back to the demand for good i we can write

$$\frac{p_i}{\lambda} = \left(\frac{y_i}{\overline{Y}} \right)^{\frac{-1}{\varepsilon}}$$

after noting that optimality will have the constraint satisfied with equality. We can then use this to obtain a price index

$$\begin{aligned} \overline{Y} &= \left(\int_0^1 y_i^{\frac{\varepsilon-1}{\varepsilon}} di \right)^{\frac{\varepsilon}{\varepsilon-1}} \\ 1 &= \int_0^1 \left(\left(\frac{y_i}{\overline{Y}} \right)^{\frac{-1}{\varepsilon}} \right)^{1-\varepsilon} di \\ 1 &= \int_0^1 \left(\frac{p_i}{\lambda} \right)^{1-\varepsilon} di \end{aligned}$$

[11] A better definition has the elasticity be the change in relative demand in response to changes in relative prices: $\varepsilon = \frac{\partial \log y_i/y_j}{\partial \log p_i/p_j}$.

$$\lambda = \left(\int_0^1 p_i^{1-\varepsilon} di \right)^{\frac{1}{1-\varepsilon}}$$

This gives us an interpretation of $\lambda = \overline{P}$, the (ideal) price index of the composite good $\overline{Y}$, or the price of a unit of utility (also obtained as such from the envelope theorem).

Further, notice that

$$\begin{aligned}
\overline{P}\,\overline{Y} &= \int_0^1 p_i y_i di \\
&= \int_0^1 \overline{P} \left(\frac{y_i}{\overline{Y}} \right)^{\frac{-1}{\varepsilon}} y_i di \\
&= \frac{\overline{P}}{\overline{Y}^{\frac{-1}{\varepsilon}}} \int_0^1 (y_i)^{\frac{\varepsilon-1}{\varepsilon}} di \\
&= \frac{\overline{P}}{\overline{Y}^{\frac{-1}{\varepsilon}}} \overline{Y}^{1-\frac{1}{\varepsilon}} \\
&= \overline{P}\,\overline{Y}
\end{aligned}$$

This property also follows directly from the aggregator having constant returns to scale, so that the cost function has to be linear.

So, what does this imply for the demand faced by an individual producer? Returning to the first order condition we can now write the inverse demand curve of any given variety i as a function of aggregate prices and quantities:

$$p_i = \left(\frac{y_i}{\overline{Y}} \right)^{\frac{-1}{\varepsilon}} \overline{P} \qquad \text{or} \qquad y_i = \left(\frac{p_i}{\overline{P}} \right)^{-\varepsilon} \overline{Y}.$$

Recall that the actions of no individual producer can affect $\overline{Y}$ or $\overline{P}$, so this equation is all the producer needs to solve its profit maximization problem. We can do this using the results from the previous section by noting that the elasticity of demand is

$$\varepsilon_i = -\left(\frac{\partial \log p_i}{\partial \log y_i} \right)^{-1} = \varepsilon,$$

so that all producers face the same elasticity of demand (equal to the elasticity of substitution across varieties). The markup is therefore constant across firms:

$$\mu = \frac{1}{1 - \frac{1}{\varepsilon}} = \frac{\varepsilon}{\varepsilon - 1},$$

and profits are

$$\pi_i = (\mu - 1)\, c_i\, y_i.$$

We can further solve for the price index in terms of the marginal costs $c_i = C_i'(y_i)$:

$$\overline{P} = \underbrace{\frac{\varepsilon}{\varepsilon - 1}}_{\text{Agg. Markup}} \times \underbrace{\left(\int_0^1 c_i^{1-\varepsilon} di \right)^{\frac{1}{1-\varepsilon}}}_{\text{Agg. Mrg. Cost } (\overline{C})}.$$

Bertrand and Cournot in monopolistic competition. We can also get this result directly, which also allow us to see the curvature in the firm's problem. In the case of Bertrand competition

$$\max \; p_i y_i - C_i(y_i) = \max_{p_i} \left(\frac{p_i}{\overline{P}} \right)^{1-\varepsilon} (\overline{P}\,\overline{Y}) - C_i\left(\left(\frac{p_i}{\overline{P}} \right)^{-\varepsilon} \overline{Y} \right).$$

The first order condition gives

$$\begin{aligned}
(1-\varepsilon)\, p_i^{-\varepsilon} \left(\frac{1}{\overline{P}} \right)^{1-\varepsilon} (\overline{P}\,\overline{Y}) + \varepsilon\, p_i^{-\varepsilon-1} C_i' \left(\left(\frac{p_i}{\overline{P}} \right)^{-\varepsilon} \overline{Y} \right) \left(\frac{1}{\overline{P}} \right)^{-\varepsilon} \overline{Y} &= 0 \\
(1-\varepsilon) + \varepsilon \frac{1}{p_i} C_i'(y_i) &= 0 \\
p_i &= \frac{\varepsilon}{\varepsilon - 1} C_i'(y_i)
\end{aligned}$$

In the case of Cournot competition

$$\max \; p_i y_i - C_i(y_i) = \max_{y_i} \left(\frac{y_i}{\overline{Y}} \right)^{\frac{\varepsilon-1}{\varepsilon}} (\overline{P}\,\overline{Y}) - C_i(y_i).$$

The first order condition gives

$$\begin{aligned}
\frac{\varepsilon - 1}{\varepsilon} y_i^{\frac{\varepsilon-1}{\varepsilon}-1} \left(\frac{1}{\overline{Y}} \right)^{\frac{\varepsilon-1}{\varepsilon}} (\overline{P}\,\overline{Y}) - C_i'(y_i) &= 0 \\
\frac{\varepsilon - 1}{\varepsilon} \left(\frac{y_i}{\overline{Y}} \right)^{-\frac{1}{\varepsilon}} \overline{P} - C_i'(y_i) &= 0 \\
p_i &= \frac{\varepsilon}{\varepsilon - 1} C_i'(y_i)
\end{aligned}$$

This establishes that under monopolistic competition with constant elasticity of demand Bertrand and Cournot competition are equivalent.

Firms' production scale. Markups are constant and so the differences between firms come from differences in marginal costs. More productive firms have lower costs and therefore charge lower prices. This translates into more productive firms operating at a larger scale, determined by the solution to

$$\left(\frac{y_i}{\overline{Y}}\right)^{\frac{-1}{\varepsilon}} \overline{P} = \frac{\varepsilon}{\varepsilon - 1} C_i'\left(y_i\right).$$

In the special case of constant marginal costs and replacing by the aggregate price we have

$$y_i = \left(\frac{c_i}{\overline{C}}\right)^{-\varepsilon} \overline{Y},$$

so that firms with lower marginal costs produce more, with elasticity ε of equilibrium output relative to marginal costs.

The scale of the problem is given by $\overline{Y}$ that is for now taken as given. In a complete equilibrium model $\overline{P}\,\overline{Y}$ must equal the income of the household (if there are no other expenses in the economy).

15.1. Taste for variety

The original use of this model as well as many current uses in growth and trade theory has to do with the concept of taste for variety. The idea is that individuals like having choice. This is captured by allowing there to be more varieties of goods.

Let there be (a continuum of) N goods and assume that all the firms are identical and have a constant marginal cost of $c_i = c$. Then, all firms set the same price

$$p_i = \frac{\varepsilon}{\varepsilon - 1} c.$$

Even though all firms set the same price, the aggregate price does not satisfy $\overline{P} = p_i$. This comes from the fact that there are now N varieties and so

$$\overline{Y} = \left(\int_0^N y_i^{\frac{\varepsilon-1}{\varepsilon}} di\right)^{\frac{\varepsilon}{\varepsilon-1}} \qquad \text{and} \qquad \overline{P} = \left(\int_0^N p_i^{1-\varepsilon} di\right)^{\frac{1}{1-\varepsilon}}$$

The price index is now

$$\overline{P} = N^{\frac{-1}{\varepsilon-1}} \frac{\varepsilon}{\varepsilon - 1} c$$

which is decreasing in the number of varieties! So, the more varieties there are the lower the price per unit of utility (or output) despite the price of each variety being the same.

This is the effect of the taste of variety.

This effect of varieties over the price index has implications for individual firms. Their output depends on the number of varieties

$$y_i = N^{\frac{-\varepsilon}{\varepsilon-1}} \overline{Y},$$

as do their profits

$$\pi_i = p_i y_i - c y_i = \frac{1}{\varepsilon - 1} c y_i = \frac{1}{\varepsilon - 1} c N^{\frac{-\varepsilon}{\varepsilon-1}} \overline{Y}.$$

Whether output and profits are increasing or decreasing in N depends on the effect of N on total demand $\left(\overline{Y}\right)$. When embedding this problem into a more complete model the level of utility (or output) $\overline{Y}$ will also respond to the number of varieties as the price per unit $\left(\overline{P}\right)$ decreases. This is a pecuniary price externality that works through changes in the aggregate price $\overline{P}$: higher N lower $\overline{P}$ and that has an effect on $\overline{Y}$.

Crucially, individual firms take the number of varieties as given. In particular, when making entry decisions, firms do not take into account their effect on the profits of other firms or on welfare. This means that the taste for variety introduces an externality into the model and so the equilibrium solution is in general not efficient.

We can see this better by adding one assumption. There is a total amount of income I to be spent in goods. If the amount of expenditure is constant that means that the aggregate demand for goods $\left(\overline{Y}\right)$ has unit elasticity. So we know that

$$I = \int_0^N p_i y_i di = \overline{P}\,\overline{Y},$$

so that as $\overline{P}$ decreases, $\overline{Y}$ must increase. Under this fixed-expenditure assumption we get to set the scale of $\overline{Y}$ as

$$\overline{Y} = \left(1 - \frac{1}{\varepsilon}\right) \frac{I}{c} N^{\frac{1}{\varepsilon-1}}.$$

Replacing into the output of individuals firms this implies

$$y_i = \left(1 - \frac{1}{\varepsilon}\right) \frac{I}{c} N^{-1}$$

so that output and profits decrease with variety. This result is, of course, not general. For most problems aggregate demand is more than unit-elastic and so total expenditure increases when the price decreases.

Equilibrium and entry. Consider an economy with an infinite number of firms that produce differentiated goods using a linear technology in labor and that differ in their productivity, as captured by their constant marginal costs, $c_i = w/z_i$ with w the market wage. The index of the firm reflects their marginal costs and so $c_i < c_j$ if and only if $i < j$. Opening a firm to produce has a fixed cost of $\psi \geq 0$. The demand for differentiated goods comes from a representative agent with preferences for the composite good

$$\overline{Y} = \left(\int_0^{\infty} y_i^{\frac{\varepsilon-1}{\varepsilon}} di \right)^{\frac{\varepsilon}{\varepsilon-1}}$$

captured by the demand curve $\overline{Y} = \overline{P}^{-\eta}$ with constant elasticity of demand η.

There are three stages in the economy.

(a) Potential entrants simultaneously decide whether or not to enter and pay the fixed cost $\psi > 0$. Let N be the number of firms that enter to produce.

(b) Each entrant i simultaneously choose prices p_i.

(c) Consumers observe prices of the differentiated goods s and make consumption decisions. Let y_i be the demand of entrant i.

We focus on sub-game perfect equilibria. An equilibrium is an entry decision for each firm i, a price function $\{p_i\}$, and a demand function $\{y_i\}$ such that

(a) Entry is profit maximizing, so that firms enter if and only if they make non-negative profits after taking into account fixed costs.

(b) Firms set prices to maximize profits taking as given the prices of other firms and the demand for their variety.

(c) The demand for each variety is chosen to minimize costs of the consumer taking as given the prices of the consumers.

(d) Total demand for the composite good satisfies the demand $\overline{Y} = \overline{P}^{-\eta}$.

We already know the price setting behavior of firms. Their profits are non-decreasing in their index as higher i corresponds to (weakly) lower marginal costs. That is, we want an index N such that

$$\pi_N(N) = (\mu - 1) c_N y_N = \psi \longrightarrow y_N = \frac{\psi}{(\mu - 1) c_N}$$

The equilibrium is given by the value of N that solves this equation.

We can go further by recalling the demand function

$$y_N = \left(\frac{p_N}{\overline{P}}\right)^{-\varepsilon} \overline{Y} = \left(\mu \frac{c_N}{\overline{P}}\right)^{-\varepsilon} \overline{P}^{-\eta}.$$

We also know that

$$\overline{P} = N^{\frac{-1}{\varepsilon-1}} \mu \overline{C},$$

where $\overline{C}$ is the average marginal cost (already divided by the number of firms to cancel variety effects). Replacing gives

$$N^{\frac{\eta-\varepsilon}{\varepsilon-1}} \left(\frac{c_N}{\overline{C}}\right)^{-\varepsilon} \left(\mu \overline{C}\right)^{-\eta} = \frac{\psi}{(\mu - 1)\, c_N}$$

In the case of constant marginal costs we further have

$$N = \left(\mu^{\eta} c^{\eta-1} \frac{\psi}{\mu - 1}\right)^{\frac{\varepsilon-1}{\eta-\varepsilon}}.$$

Efficiency of equilibrium. Lets focus on the case treated in Dixit and Stiglitz (1977). Firms are symmetric with constant marginal cost c. There is some fixed income I to be spent across varieties, so the (aggregate) demand for goods has an elasticity of one ($\eta = 1$).

In this case we have (from symmetry and the fixed income) that

$$p_i = p = \mu c \qquad \text{and} \qquad y_i = y = \frac{I}{N p} = \frac{I}{N \mu c}.$$

We also have (from aggregation)

$$\overline{Y} = N^{\frac{\varepsilon}{\varepsilon-1}} y \qquad \text{and} \qquad \overline{P} = N^{\frac{1}{1-\varepsilon}} p.$$

Free entry implies the equilibrium number of varieties

$$\begin{aligned}
(p - c)\, y &= \psi \\
\left(1 - \frac{1}{\mu}\right) \frac{I}{N} &= \psi \\
\left(1 - \frac{1}{\mu}\right) \frac{I}{\psi} &= N^{\star} \\
\frac{1}{\varepsilon} \frac{I}{\psi} &= N^{\star}
\end{aligned}$$

and thus the output of each firm

$$y^\star = (\varepsilon - 1)\frac{\psi}{c}.$$

We now compare this solution to a constrained social optimum, where the allocation must have firm make non-negative profits. The unconstrained optimum can have firm price below average cost (because of the presence of fixed costs) and therefore implicitly requires lump-sum transfers to firms.

The constrained optimum should have all active firms be identical (as there are no technology- or preference-based differences between them). They also must be making exactly zero profits. The problem is to choose the number of firms as well as firm quantities and prices to maximize $\overline{Y}$ (the utility level). The resource constraint implies that $\overline{Y} = I/\overline{P}$, and so, with fixed income, we know that the utility is maximized as we minimize aggregate prices.

From the definition of the price index and the symmetry across firms we can then express the problem as one of minimizing the price index subject to the zero profit condition of firms

$$\min_{p,N} N^{\frac{1}{1-\varepsilon}} p \qquad \text{s.t. } (p-c)\frac{I}{Np} = \psi.$$

The first order conditions are

$$N^{\frac{1}{1-\varepsilon}} = \frac{c}{p^2}\frac{I}{N}\lambda$$
$$\frac{1}{\varepsilon - 1}N^{\frac{1}{1-\varepsilon}-1}p = (p-c)\frac{I}{N^2 p}\lambda$$

The ratio between them gives

$$p = \frac{\varepsilon}{\varepsilon - 1}c.$$

This is the same optimal pricing choice as in the decentralized equilibrium. The number of firms is also the same as in the market's solution. Therefore the solution is constrained-efficient.

The unconstrained equilibrium is not subject to the zero-profit condition, but instead must acknowledge the effect of fixed costs on disposable income (now that firm profits need not cover them). The solution still requires firms to behave symmetrically. The problem is then to maximize $\overline{Y}$ subject to resources

$$\max_{y,N} N^{\frac{\varepsilon}{\varepsilon-1}} y \qquad \text{s.t. } I = N(cy + \psi)$$

The first order conditions are

$$N^{\frac{\varepsilon}{\varepsilon-1}} = Nc\lambda$$
$$\frac{\varepsilon}{\varepsilon-1} N^{\frac{\varepsilon}{\varepsilon-1}-1} y = (cy + \psi)\lambda$$

The ratio between them gives the unconstrained optimum's output per firm:

$$y = (\varepsilon - 1)\frac{\psi}{c}.$$

We can get the number of active firms from the constraint:

$$N = \frac{1}{\varepsilon}\frac{I}{\psi}.$$

These are the same number of varieties and output per firm as in the equilibrium. However, this turns out to be a knife-edge case in which the equilibrium is efficient due to the fixed income (expenditure) assumption. Dixit and Stiglitz (1977) show that the unconstrained number of varieties is larger than the equilibrium level when the expenditure in the differentiated goods responds to their (aggregate) price.

16. Variable Markups

We now study another dimension of firm heterogeneity: variable markups. Firms of different sizes can differ in the markups they charge. This is not the case in the monopolistic competition framework above because of two reasons. First, the demand faced by each producer had constant elasticity, and so the markup was constant. Recall that optimal markups are $\mu = 1/\left(1-\frac{1}{\varepsilon}\right)$. Second, there were no strategic interactions between firms. The number of firms (or products) was assumed to be large enough to make no firm have an effect on the aggregate. If firms are large enough to internalize their effect on others larger firms gain market power and can thus charge larger markups.

The objective in this section is to go over the basics of these two approaches. Each has extensive applications in the economics of misallocation, monetary policy, and international trade.

16.1. Variable Elasticity of Demand and Variable Markups

We start extending the framework of Dixit and Stiglitz (1977) to allow for a demand system with variable elasticity of demand. In this we follow Kimball (1995).[12]

In order to have variable elasticity of demand we need a new formulation of the aggregator. There is an aggregate (or composite) good Y that is produced by a competitive producer. Production combines a continuum of differentiated goods $\{y_i\}$. The aggregation technology is implicitly defined by

$$1 = \int \Upsilon\left(\frac{y_i}{Y}\right) di.$$

The properties of the demand for the differentiated goods y_i depend on the function Υ. The function Υ is either strictly increasing and strictly concave (case where the goods are gross substitutes) or strictly decreasing and strictly convex (case where the goods are gross complements). Note that the function is homogeneous of degree 1 by construction.

This function includes the constant-elasticity-of-substitution (CES) aggregator used above when we let $\Upsilon(x) = x^{\frac{\varepsilon-1}{\varepsilon}}$. Other functional forms allow for variable-elasticity of demand. To see this more clearly we need to solve the problem of the aggregator.

The problem of the aggregator is to minimize its cost subject to a minimum level of output $\overline{Y}$:

$$\min_{\{y_i\}} \int p_i y_i di \qquad \text{s.t. } 1 = \int \Upsilon\left(\frac{y_i}{\overline{Y}}\right) di$$

The first order conditions are (for each variety)

$$p_i = \Upsilon'\left(\frac{y_i}{Y}\right)\frac{\lambda}{Y},$$

where λ is the Lagrange multiplier. We need to solve for the multiplier. To do this, multiply both sides of the first order condition by y_i and sum across varieties to obtain

$$\lambda = \frac{\int p_i y_i di}{\int \Upsilon'\left(\frac{y_i}{Y}\right)\frac{y_i}{Y} di}.$$

Then, define the price of the aggregate good, P, so that it satisfies $P \cdot Y = \int p_i \cdot y_i di$. Finally, replace λ into the first order condition and divide both sides by P to obtain the inverse

[12] Other formulations are reviewed by Costas Arkolakis and Monica Morlacco, see their notes on variable elasticity of demand.

demand curve for variety i expressed in terms of relative prices and relative output:

$$\frac{p_i}{P} = \frac{\Upsilon'\left(\frac{y_i}{Y}\right)}{\int \Upsilon'\left(\frac{y_i}{Y}\right)\frac{y_i}{Y}di}.$$

We can also define implicitly the aggregate price. For this it is convenient to denote $D = \int \Upsilon'\left(\frac{y_i}{Y}\right)\frac{y_i}{Y}di$ and then we define a set of two equations that jointly determine D and P using the definition of the price index and the aggregator:

$$P = \int p_i \cdot \left(\Upsilon'\right)^{-1}\left(D\frac{p_i}{P}\right)di \qquad \text{and} \qquad 1 = \int \Upsilon\left(\left(\Upsilon'\right)^{-1}\left(D\frac{p_i}{P}\right)\right)di,$$

where $\left(\Upsilon'\right)^{-1}(x)$ is the inverse function of the first derivative of Υ.

While this looks complicated the key is in realizing that the behavior of the monopolists producing each variety depends only on the elasticity of demand $\varepsilon_i = -\left(\frac{\partial \log p_i}{\partial \log y_i}\right)^{-1}$. As before, the key is that the actions of a single producer cannot affect the value of aggregate output or prices. The elasticity is

$$\begin{aligned}
\varepsilon_i &= -\left(\frac{y_i}{p_i}\frac{\partial p_i}{\partial y_i}\right)^{-1} \\
&= -\left(\frac{y_i}{p_i}\frac{\partial \Upsilon'\left(\frac{y_i}{Y}\right)\frac{P}{D}}{\partial y_i}\right)^{-1} \\
&= -\left(\frac{y_i}{p_i}\Upsilon''\left(\frac{y_i}{Y}\right)\frac{P}{D}\frac{1}{Y}\right)^{-1} \\
&= -\left(\frac{y_i}{Y}\right)^{-1}\frac{\Upsilon'\left(\frac{y_i}{Y}\right)}{\Upsilon''\left(\frac{y_i}{Y}\right)}
\end{aligned}$$

So, knowing the properties of the first two derivatives of Υ is enough to know the behavior of firms.

When goods are substitutes we have $\Upsilon'(x) > 0$ and $\Upsilon''(x) < 0$. When they are complements we have $\Upsilon'(x) < 0$ and $\Upsilon''(x) > 0$. So we know that the elasticity is positive (recall we fixed the sign, the demand curve is downward sloping). The key question is whether it is increasing or decreasing in the variety's (relative) output.

In practice, we work with aggregators that imply a decreasing elasticity of demand for larger firms, capturing the fact that larger firms have more market power and charge a higher markup. For example, the aggregator proposed by Klenow and Willis (2016) implies

that

$$\varepsilon_i = \varepsilon \cdot \left(\frac{y_i}{Y}\right)^{-\frac{\theta}{\varepsilon}}$$

with $\varepsilon, \theta > 0$ parameters, so that as the relative output increases the elasticity decreases.

In equilibrium, more productive firms have lower marginal costs and hence can charge lower prices. This leads them to operate in more inelastic portion of their demand curves and charge higher markups. So, higher markups coexist with lower prices.

16.2. Oligopolistic Competition and Variable Markups

Finally, we consider the case of oligopolistic competition, where finitely many firms compete à la Cournot or à la Bertrand in a market. In this setup, it is the strategic interactions between firms that give rise to market power for the more productive firms who end up being larger in equilibrium. These firms charge higher markups. This setup is developed in Atkeson and Burstein (2008) and used in a variety of papers in international trade and misallocation.

Consider a market with N producers of a differentiated goods competing à la Cournot (so monopolists choose quantities taking quantities of other monopolists as given). Firms are heterogeneous in their marginal cost, c_i. The demand for the goods comes from a competitive final good producer that aggregates individual goods using a CES technology

$$Y = \left(\sum_{i=1}^{N} y_i^{\frac{\varepsilon-1}{\varepsilon}}\right)^{\frac{\varepsilon}{\varepsilon-1}}.$$

The final good producer is a price taker and the demand for final goods has constant elasticity of demand η, so that $Y = P^{-\eta}$. We assume that $\eta < \varepsilon$. This means that the varieties are more substitutable between them than the total output is for other (potential) goods that can be in the economy. This is the case when we have a model with many markets, being aggregated with an (outer) CES aggregator with elasticity η.

Just as before, the cost minimization problem of the final good producer implies a demand for variety i of

$$\frac{p_i}{P} = \left(\frac{y_i}{Y}\right)^{-\frac{1}{\varepsilon}},$$

where

$$PY = \sum_{i=1}^{N} p_i y_i \qquad \text{and} \qquad P = \left(\sum_{i=1}^{N} p_i^{1-\varepsilon}\right)^{\frac{1}{1-\varepsilon}}.$$

What changes is the behavior of the monopolists. The key is that the elasticity of demand they face depends on how their output affects aggregates. We now have

$$\varepsilon_i^{-1} = -\frac{\partial \log p_i}{\partial \log y_i} = -y_i \frac{\partial \log \left(\frac{y_i}{Y}\right)^{\frac{-1}{\varepsilon}} P}{\partial y_i} = -y_i \frac{\partial \log \left(\frac{y_i}{Y}\right)^{-\frac{1}{\varepsilon}} Y^{-\frac{1}{\eta}}}{\partial y_i} = \frac{\partial \log y_i^{\frac{-1}{\varepsilon}} Y^{\frac{1}{\varepsilon}-\frac{1}{\eta}}}{\partial y_i},$$

so that we can break the elasticity into two terms, the effect of a change in the variety's elasticity and the effect on aggregate production,

$$\varepsilon_i^{-1} = \underbrace{\frac{1}{\varepsilon}}_{\text{Elast. of Subs.}} + \left(\frac{1}{\eta} - \frac{1}{\varepsilon}\right) \underbrace{\frac{y_i}{Y}\frac{\partial Y}{\partial y_i}}_{\text{Elast. of Agg. Output}}.$$

The elasticity of the aggregate output is with respect to variety i is

$$\frac{y_i}{Y}\frac{\partial Y}{\partial y_i} = \frac{y_i}{Y}\frac{\partial \left(\sum_{i=1}^N y_i^{\frac{\varepsilon-1}{\varepsilon}}\right)^{\frac{\varepsilon}{\varepsilon-1}}}{\partial y_i} = \frac{y_i}{Y}\left(\sum_{i=1}^N y_i^{\frac{\varepsilon-1}{\varepsilon}}\right)^{\frac{\varepsilon}{\varepsilon-1}-1} y_i^{\frac{\varepsilon-1}{\varepsilon}-1} = \frac{y_i^{\frac{\varepsilon-1}{\varepsilon}}}{\sum_{i=1}^N y_i^{\frac{\varepsilon-1}{\varepsilon}}} = \left(\frac{y_i}{Y}\right)^{1-\frac{1}{\varepsilon}}$$

This gives us the result. But if we observe this expression in detail we can see that we can also express it as

$$\frac{y_i}{Y}\frac{\partial Y}{\partial y_i} = \left(\frac{y_i}{Y}\right)^{-\frac{1}{\varepsilon}} \frac{y_i}{Y} = \frac{p_i y_i}{PY} = s_i$$

which is the sales-share of producer i. This is a key result. The elasticity of demand is increasing in the firm's market share. The sign follows from the assumption that $\eta < \varepsilon$.

Replacing back we get that the demand elasticity of variety i is a weighted average between the elasticity of its own variety and the demand elasticity of the market:

$$\frac{1}{\varepsilon_i} = \frac{1}{\varepsilon}\left(1 - s_i\right) + \frac{1}{\eta} s_i.$$

If the firm is small $\left(s_i \to 0\right)$, it behaves like in the monopolistic competition setup and only cares about the elasticity of its own variety, taking aggregates as given. As the firm grows large $\left(s_i \to 1\right)$, it behaves like a true monopolist, caring about the elasticity of demand for the market output (Y) and not for the competition with other varieties.

The markup is then

$$\mu_i = \frac{1}{1 - \frac{1}{\varepsilon_i}} = \frac{1}{\left(1 - \frac{1}{\varepsilon}\right)\left(1 - s_i\right) + \left(1 - \frac{1}{\eta}\right) s_i}.$$

It is again the case that larger firms have larger markups, reflecting the decrease in elasticity as market shares increase.

Aggregating markups. The average markup in a market, $\bar{\mu}$, is defined as the ratio between the market's price P and the market's marginal cost $\bar{c}$,

$$\bar{\mu} = \frac{P}{\bar{c}}.$$

This is equivalent to defining it as the ratio of the market's revenue PY and the market's total cost, which, under constant returns-to-scale, is $\sum_i c_i y_i$.

Because the market aggregator has constant-returns-to-scale, the market's marginal cost is equal to the output-weighted average of the individual marginal costs, that we label $c_i \equiv C_i'(y_i)$,

$$\bar{c} = \sum_{i=1}^{N} c_i \frac{y_i}{Y}.$$

Then, the market's markup is obtained as a sales-weighted harmonic mean of individual markups

$$\bar{\mu} = \frac{P}{\bar{c}} = \left[\sum_{i=1}^{N} c_i \frac{y_i}{PY}\right]^{-1} = \left[\sum_{i=1}^{N} \frac{1}{\mu_i} s_i\right]^{-1} = \left[\sum_{i=1}^{N_m} \left(1 - \frac{1}{\varepsilon_i}\right) s_i\right]^{-1} = \frac{1}{1 - \frac{1}{\bar{\varepsilon}}},$$

where $\bar{\varepsilon}$ is the (weighted-harmonic) average elasticity in the market,

$$\bar{\varepsilon} \equiv \left[\sum_{i=1}^{N} \frac{1}{\varepsilon_i} s_i\right]^{-1}.$$

In this way the market's average markup has the same expression as the individual markup with the corresponding market elasticity.

In the case of Cournot competition we can go further using the expression derived above for the elasticity of each firm:

$$\frac{1}{\bar{\varepsilon}} = \sum_{i=1}^{N} \frac{1}{\varepsilon_i} s_i = \sum_{i=1}^{N} \left(\frac{1}{\varepsilon}(1 - s_i) + \frac{1}{\eta} s_i\right) s_i = \frac{1}{\varepsilon}\left(\sum_{i=1}^{N} s_i - \sum_{i=1}^{N} s_i^2\right) + \frac{1}{\eta} \sum_{i=1}^{N} s_i^2$$

where we take advantage of the fact that $\sum s_i = 1$ and that the sum of square shares is the definition of the Herfindahl-Hirschman index of concentration (HHI). This index gives the

probability that two random dollars spent in the market are spent in the same firm. The result is a direct link between concentration in the market and average markups

$$\frac{1}{\bar{\varepsilon}} = \frac{1}{\varepsilon}(1 - \text{HHI}) + \frac{1}{\eta}\text{HHI} \qquad \text{and} \qquad \frac{1}{\bar{\mu}} = \underbrace{\frac{\eta - 1}{\eta}}_{\text{Monopoly Markup}} + \underbrace{\left(\frac{1}{\eta} - \frac{1}{\varepsilon}\right)(1 - \text{HHI})}_{\text{Concentration Markup}}.$$

17. Aggregation and Misallocation

Once we have a theory of firm heterogeneity we can start asking questions about misallocation of resources and hence of economic activity. The question is the same of the welfare theorems: is the market equilibrium efficient? Here we will focus on efficiency in terms of production. The exposition here follows the review of the literature in Hopenhayn (2014).

17.1. Aggregation, and Efficiency

Lets start by revisiting the span of control model. As we have seen already, the key for firm heterogeneity is to introduce decreasing returns to scale. This can be done from the technology side as in Lucas (1978a) or from the demand side as in Dixit and Stiglitz (1977). Both setups deliver similar results and so we start with the span of control problem in its simpler form. We show that both setups are efficient.

There is a population of N workers with preferences for good consumption (no disutility from labor). There is a mass M of firms that differ in their productivity distributed according to Γ. The firms' production technology uses only labor and is

$$y_i = z_i n_i^{\alpha}.$$

There are no operational costs.

The notation changed relative to the one above to make results more comparable, so that y_i is the quantity produced by firm i.

Market Economy. Markets are competitive. The wage rate is w, the price of the final good is normalized to 1. We know that the optimal size is

$$n^{\star}(z) = \left(\frac{\alpha z}{w}\right)^{\frac{1}{1-\alpha}} \qquad \text{and} \qquad y^{\star}(z) = z^{\frac{1}{1-\alpha}}\left(\frac{\alpha}{w}\right)^{\frac{\alpha}{1-\alpha}}.$$

To clear the labor market we need that

$$N = \int_0^M n^\star\left(z_i\right) di = \left(\frac{\alpha}{w^\star}\right)^{\frac{1}{1-\alpha}} \int_0^M z_i^{\frac{1}{1-\alpha}} di,$$

so the equilibrium wage is

$$w^\star = \alpha \left(\int_0^M z_i^{\frac{1}{1-\alpha}} di\right)^{1-\alpha} N^{\alpha-1}.$$

Thus, the equilibrium firm size is

$$y^\star\left(z_i\right) = z_i^{\frac{1}{1-\alpha}} \left(\frac{N}{\int_0^M z_j^{\frac{1}{1-\alpha}} dj}\right)^{\alpha}$$

and total output is

$$Y = \int_0^M y^\star\left(z_i\right) di = \underbrace{\left(\int_0^M z_i^{\frac{1}{1-\alpha}} di\right)^{1-\alpha}}_{\text{Agg. Productivity}} N^{\alpha}.$$

Therefore, this economy has an aggregate production function that looks the same as the individual production function of a firm but with an aggregate productivity given by the distribution of firms. This production function has decreasing returns to scale if we take as firms as a fixed production factor.

We can get a different and more informative expression if we focus on the distribution of firm productivity and separate from the number of firms:

$$Y = \overbrace{Z}^{\text{Avg. Prod.}} \underbrace{M^{1-\alpha}}_{\text{Firm Scale}} N^{\alpha}.$$

Here, Z is the average productivity across firms in the economy, we define it as

$$Z = \left(E\left[z_i^{\frac{1}{1-\alpha}}\right]\right)^{1-\alpha} = \left(\int z^{\frac{1}{1-\alpha}} d\Gamma\left(z\right)\right)^{1-\alpha}.$$

Seen this way, aggregate production has constant returns to scale, and is increasing in the number of firms. This is, unsurprisingly, the same result as in the taste for variety explored above. Another important result is that the decreasing returns to scale make the harmonic mean the way to aggregate

Efficient Allocation. The planner's objective is to maximize output as that maximizes utility, so,

$$\max_{\{n_i\}} \int_0^M z_i n_i^{\alpha} di \qquad \text{s.t. } N = \int_0^M n_i di.$$

The market economy is efficient. To see this, note that the efficient allocation equates average products across firms

$$\alpha z_i n_i^{\alpha-1} = \lambda$$
$$\frac{y_i}{n_i} = \frac{\lambda}{\alpha}$$

This already happened in the market solution with $\lambda = w$. Hence, firm size and the aggregate results above hold unchanged in the efficient allocation. **There is no misallocation in this economy**.

Connection to The Monopolistic Competition Case. Under monopolistic competition we have constant returns to scale technology $\left(y_i = z_i n_i\right)$ but curvature in demand (with constant elasticity) we know that the market solution has

$$y_i = \left(\frac{c_i}{\left(\int_0^M c_i^{1-\varepsilon} di\right)^{\frac{1}{1-\varepsilon}}} \right)^{-\varepsilon} Y = \left(\frac{z_i}{\left(\int_0^M z_i^{\varepsilon-1} di\right)^{\frac{1}{\varepsilon-1}}} \right)^{\varepsilon} Y,$$

To clear the labor market the wage must be such that

$$N = \int_0^M n_i di = \frac{y_i}{z_i} di = \int_0^M z_i^{\varepsilon-1} \left(\int_0^M z_i^{\varepsilon-1} di \right)^{\frac{-\varepsilon}{\varepsilon-1}} Y di = \left(\int_0^M z_i^{\varepsilon-1} di \right)^{\frac{-1}{\varepsilon-1}} Y$$

Therefore, aggregate output (or utility) is

$$Y = \overbrace{Z}^{\text{Avg. Prod.}} \underbrace{M^{\frac{1}{\varepsilon-1}}}_{\text{Taste for Variety}} N,$$

where $Z = \left(E\left[z^{\varepsilon-1}\right]\right)^{\frac{1}{\varepsilon-1}} = \left(\int z^{\varepsilon-1} d\Gamma(z)\right)^{\frac{1}{\varepsilon-1}}$ is the average productivity.

This is the same result we had above once we adjust for the curvature of the problem. Notice that aggregation is linear (as above) for a monotone transformation of output $\tilde{Y} = Y^{\frac{\varepsilon-1}{\varepsilon}}$ and $\tilde{y}_i = y_i^{\frac{\varepsilon-1}{\varepsilon}}$. In this case the aggregation gives the same result as in the span of

control model

$$\tilde{Y} = \overbrace{\tilde{Z}}^{\text{Avg. Prod.}} \underbrace{M^{\frac{1}{\varepsilon}}}_{\text{Taste for Variety}} N^{\frac{\varepsilon-1}{\varepsilon}}$$

with $\alpha = \frac{\varepsilon-1}{\varepsilon}$ and $\tilde{Z} = Z^{\frac{\varepsilon-1}{\varepsilon}}$.

We already know that entry is (constrained) efficient in this setup.

The planner's problem is (for the monotone transformation) is

$$\max_{\{n_i\}} \underbrace{\int_0^M \left(z_i n_i\right)^{\frac{\varepsilon-1}{\varepsilon}} di}_{\tilde{Y}} \qquad \text{s.t. } N = \int_0^M n_i di.$$

Optimality requires

$$\frac{\varepsilon-1}{\varepsilon}\left(z_i\right)^{\frac{\varepsilon-1}{\varepsilon}} n_i^{\frac{\varepsilon-1}{\varepsilon}-1} = \lambda$$

$$\frac{\tilde{y}_i}{n_i} = \frac{\varepsilon}{\varepsilon-1}\lambda$$

so all firms have the same average product (as before). This condition is satisfied in equilibrium because the pricing choice implies

$$p_i = \frac{\varepsilon}{\varepsilon-1}\frac{w}{z_i}$$

$$\frac{y_i^{\frac{\varepsilon-1}{\varepsilon}}}{n_i} = \frac{\varepsilon}{\varepsilon-1}\frac{w}{P}Y^{\frac{1}{\varepsilon}}$$

$$\frac{\tilde{y}_i}{n_i} = \frac{\varepsilon}{\varepsilon-1}\frac{w}{P}Y^{\frac{1}{\varepsilon}}$$

So, the market solution is efficient as average products are dutifully equalized across firms.

17.2. Wedges and Misallocation

The key for the efficiency of the models above is that firms in the market economy had the same marginal product of labor as they are all responding to the same costs (in the case of the examples above, the same wage rate). Equating the marginal products across firms is precisely what you need if you want to maximize output. This is not the case in the data. The discrepancies come from firm-specific frictions that distort the firm problem. For example, some firms can be subject to certain forms of taxation while some others are not (or just evade taxes).

Example: Labor wedge. Consider an example with firm-specific labor taxes (or equivalently, additional labor costs that vary across firms):

$$\max z_i n_i^{\alpha} - \left(1+\tau_i\right) w.$$

In this case the optimal labor choice and output of a firm become

$$n^{\star}\left(z_i\right) = \left(\frac{\alpha z_i}{\left(1+\tau_i\right) w}\right)^{\frac{1}{1-\alpha}} \qquad \text{and} \qquad y^{\star}\left(z_i\right) = z_i^{\frac{1}{1-\alpha}} \left(\frac{\alpha}{\left(1+\tau_i\right) w}\right)^{\frac{\alpha}{1-\alpha}}.$$

When we aggregate we can see the role of the distortions. Market clearing now requires the wage to be

$$w^{\star} = \alpha \left(\int_0^M \left(\frac{z_i}{1+\tau_i}\right)^{\frac{1}{1-\alpha}} di\right)^{1-\alpha} N^{\alpha-1},$$

and aggregate output becomes

$$Y = \left(\overbrace{\int \underbrace{\frac{\left(\frac{z_i}{1+\tau_i}\right)^{\frac{1}{1-\alpha}}}{\int \left(\frac{z_i}{1+\tau_i}\right)^{\frac{1}{1-\alpha}} d\Gamma\left(z_i\right)}}_{\omega_i \text{ Distorted Prod. Weights}} \left(1+\tau_i\right) d\Gamma\left(z_i\right)}^{\text{Weighted Distortion}}\right) \underbrace{\left(\int \left(\frac{z_i}{1+\tau_i}\right)^{\frac{1}{1-\alpha}} d\Gamma\left(z_i\right)\right)^{1-\alpha}}_{\text{Distorted Productivity}} M^{1-\alpha} N^{\alpha}$$

where the distortions affect the problem twice, once distorting productivity and then also as an average distortion weighted by productivity.

An important implication of this result is that distortions must vary across firms for them to actually distort. If $\tau_i = \tau$ we can easily see that it cancels out of the expression above.

TFPR. In the one input case studied so far (say, only labor) misallocation comes from differences in marginal product or average products. When there are several inputs this is not necessarily the case. Instead, for a production function $y_i = z_i \left(f\left(\vec{x}_i\right)\right)^{\alpha}$ with f a constant returns to scale function in inputs $\vec{x} = (x_1, \ldots, x_N)$, what must be equated across firms is the ratio

$$\frac{y_i}{f\left(\vec{x}_i\right)} = \text{TFPR}$$

This ratio is called the total factor productivity of revenue.

We can see how the TFPR determines the importance of distortions by focusing in

the one-input case again. In that case $\text{TFPR}_i = \frac{y_i}{n_i}$. The efficient (planner's) choice would have been to have $\kappa = \frac{y_i^p}{n_i^p} = \text{TFPR}^p$ and so variation in TFPR captures misallocation in this model.

We can see the effect of distortions as the ratio of the market's allocation to the planner's labor choice:

$$\theta_i = \frac{n_i}{n_i^p} = \frac{y_i^p}{n_i^p}\frac{n_i}{y_i}\frac{y_i}{y_i^p} = \frac{\kappa}{\text{TFPR}_i}\left(\frac{n_i}{n_i^p}\right)^{\alpha} = \frac{\kappa}{\text{TFPR}_i}\theta_i^{\alpha} \longrightarrow \theta_i = \left(\frac{\text{TFPR}_i}{\kappa}\right)^{\frac{-1}{1-\alpha}}$$

So the variation in TFPR goes hand in hand with the distortions to the allocation, captured by θ.

In the context of the example above with a wedge for labor costs we get $\theta_i \propto \left(1+\tau_i\right)^{\frac{1}{1-\alpha}}$, so we can obtain the distortion as function of wedges as well. The advantage of TFPR is that, unlike wedges, we can often measure it.

We can now relate the allocation distortions $\{\theta_i\}$ and the TFPR to productivity. Total output in the distorted economy is

$$Y^d = \int_0^M z_i n_i^{\alpha} di = \int_0^M \theta_i^{\alpha}\left(z_i\left(n_i^p\right)^{\alpha}\right) di = \int_0^M \theta_i^{\alpha} y_i^p di = \kappa \int_0^M \theta_i^{\alpha} n_i^p di$$

But, notice that κ is the average labor productivity in the undistorted (planner's) economy. So it must be true that $Y^p = \kappa N$. Replacing we get

$$\frac{Y^d}{Y^p} = \int_0^M \theta_i^{\alpha} \underbrace{\frac{n_i^p}{N}}_{\text{Efficient Labor Weights}} di$$

Moreover, this ratio is also equal to the ratio of TFP between the distorted and the undistorted economies because the total labor input is the same (recall from the problem above that the only difference between the aggregate production functions is in productivity). This gives

$$\frac{\text{TFP}^d}{\text{TFP}^p} = \int \theta^{\alpha} dN(\theta),$$

where $N(\theta) = \int_{\{i|\theta_i<\theta\}} \frac{n_i^p}{N} di$ is the distribution of employment in the efficient allocation of firms with distortions $\theta_i < \theta$. However, this expression still depends on unknown quantities.

This formula is nevertheless useful because it highlight an important property of distortions. In the words of Hopenhayn:

Notice that this formula is silent about the productivity of the firms underlying these distortions, so whether they are correlated is not important per se. Correlation matters for a different reason. For example, consider two groups of firms m_1 and m_2 with productivities $z_1 < z_2$ and optimal employments $n_1 < n_2$. Suppose further that optimal total employment in both groups is identical, i.e., $m_1 n_1^p = m_2 n_2^p$. Then the impact on productivity from shifting a fixed number of workers from one group to the other is the same regardless of whether they are shifted from the lower- to higher-productivity group or vice versa.

Finally, to see the role of TFPR we can write the TFP gap between the efficient and the market allocation using only measured quantities. Specifically, we can write (inverse of) the TFP gap as a curved harmonic mean of the relative TFPR of firms, weighted by their employment (or labor share). This is not obvious. We can construct the formula in steps. First, the employment share of a firm is

$$\frac{n_i}{N}\left(=\frac{wn_i}{wN}\right) = \left(1+\tau_i\right)^{-\frac{1}{1-\alpha}} \frac{z_i^{\frac{1}{1-\alpha}}}{\int_0^M \left(\frac{z_i}{1+\tau_i}\right)^{\frac{1}{1-\alpha}} di}.$$

Second, the relative TFPR is defined as

$$\frac{\text{TFPR}_i}{\overline{\text{TFPR}}} = \frac{y_i/n_i}{Y/N} = \frac{\frac{(1+\tau_i)w}{\alpha}}{\frac{Y}{N}} = \left(1+\tau_i\right)\left(\int_0^M \frac{\left(\frac{z_i}{1+\tau_i}\right)^{\frac{1}{1-\alpha}}}{\int_0^M \left(\frac{z_i}{1+\tau_i}\right)^{\frac{1}{1-\alpha}} di}\left(1+\tau_i\right) di\right)^{-1}.$$

Finally, we can bring them together to express

$$\frac{\text{TFP}^p}{\text{TFP}^d} = \left(\int_0^M \left(\frac{\text{TFPR}_i}{\overline{\text{TFPR}}}\right)^{\frac{1}{1-\alpha}} \frac{n_i}{Y} di\right)^{1-\alpha} = \frac{\left(\int_0^M z_i^{\frac{1}{1-\alpha}} di\right)^{1-\alpha}}{\left(\int_0^M \frac{\left(\frac{z_i}{1+\tau_i}\right)^{\frac{1}{1-\alpha}}}{\int_0^M \left(\frac{z_i}{1+\tau_i}\right)^{\frac{1}{1-\alpha}} di}\left(1+\tau_i\right) di\right)\left(\int_0^M \left(\frac{z_i}{1+\tau_i}\right)^{\frac{1}{1-\alpha}} di\right)^{1-\alpha}}.$$

Notice that this is the inverse to the expression above. A higher value implies more misallocation.

17.3. Application: Financial Frictions

Consider the economy described in Guvenen, Kambourov, Kuruscu, Ocampo, and Chen (2023b) that follows the derivations in Hsieh and Klenow (2009).

Entrepreneurs differ in productivity ($z_i > 0$) and produce a differentiated good according to a linear technology:

$$x_i = z_i k_i,$$

where k_i is the final good used in production by entrepreneur i at age h. The final good, Y, is produced according to a Cobb-Douglas technology:

$$Y = Q^{\alpha} L^{1-\alpha},$$

where Q is the CES composite of intermediate inputs:

$$Q = \left(\int x_i^{\mu} di \right)^{1/\mu}.$$

To distinguish Q from the unadjusted capital stock, $K = \int k_i di$, we refer to the former as the "quality-adjusted capital stock," since its level depends on the allocation of capital across entrepreneurs. Total Factor Productivity in the intermediate goods sector can be written as

$$\text{TFP}_Q = \frac{Q}{K},$$

where TFP_Q captures the extent of misallocation of capital. We write total output as a Cobb-Douglas function of K and L: $Y = \text{TFP} \cdot K^{\alpha} L^{1-\alpha}$, where $\text{TFP} \equiv \text{TFP}_Q^{\alpha}$ is the aggregate TFP of the economy.

The final good producing sector is competitive, so the profit maximization problem is

$$\max_{\{x_i\}, L} \left(\int x_i^{\mu} di \right)^{\alpha/\mu} L^{1-\alpha} - \int \left(p\left(x_i\right) \times x_i \right) di - \overline{w} L.$$

The first-order optimality conditions yield the inverse demand (price) function for each intermediate input and for the market wage:

$$p\left(x\right) = \alpha Q^{\alpha-\mu} L^{1-\alpha} x^{\mu-1} \qquad \overline{w} = \left(1-\alpha\right) Q^{\alpha} L^{-\alpha}.$$

The economy is distorted because of the existence of financial frictions in the form of collateral constraints:

$$k_i \leq \vartheta\left(z_i\right) \times a_i,$$

This makes the entrepreneurial problem be

$$\pi(a,z) = \max_{k \leq \vartheta(z)a} \left\{ p(zk) \times zk - (r+\delta)\, k \right\},$$

where δ is the depreciation rate. The price of the differentiated good can be written as $p(zk) = \mathcal{R} \times (zk)^{\mu-1}$, where $\mathcal{R} \equiv \alpha Q^{\alpha-\mu} L^{1-\alpha}$, yielding the solution

$$k(a,z) = \min \left[\left(\frac{\mu \mathcal{R} z^{\mu}}{r+\delta} \right)^{\frac{1}{1-\mu}}, \vartheta(z)\, a \right].$$

However, this makes it difficult to aggregate because the level of capital (and hence the scale of each entrepreneurial firm) depends on the distribution of wealth. Nevertheless, We can measure the effects of these distortions on aggregate TFP and output, following a large and growing literature that frames the discussion on misallocation in terms of various wedges, such as capital, labor, and output wedges. In particular, we follow Hsieh and Klenow (2009) and compute measures of misallocation for this economy.

Instead of modeling and capturing the effect of a particular distortion, or distortions, we infer the underlying distortions and wedges in the economy by studying the extent to which the marginal revenue products of capital and labor differ across firms. This is based on the insight that without any distortions, the marginal revenue products of capital and labor have to be equalized across all firms.[13]

TFP in the Q sector.. We will first focus on the intermediate goods sector. Under the alternative capital-wedge approach, the problem of each intermediate goods producer is

$$\pi_i = \max_{k_i} p\left(z_i k_i\right) z_i k_i - \left(1+\tau_i\right)(R+\delta)\, k_i \,,$$

where τ_i is a firm-specific wedge. There are no collateral constraints. There is only one input and, as a result, only one wedge can be identified.

The revenue TFP in sector Q for each firm i is

$$TFPR_{Q,i} \equiv \frac{p\left(x_i\right) x_i}{k_i} = \frac{1}{\mu}\left(1+\tau_i\right)(R+\delta).$$

[13] This is the case in the monopolistic competition models, such as in Hsieh and Klenow (2009). Alternatively, in environments like the ones in Lucas (1978b) and Restuccia and Rogerson (2008), in which firms feature decreasing returns to scale but produce the same homogeneous good, the marginal products of capital and labor have to be equalized in the non-distorted economy

The aggregate TFP in sector Q can be expressed as

$$TFP_Q \equiv \frac{Q}{K} = \left(\int_i \left(z_i \frac{\overline{TFPR_Q}}{TFPR_{Q,i}} \right)^{\frac{\mu}{1-\mu}} di \right)^{\frac{1-\mu}{\mu}},$$

where the average $TFPR_Q$ is

$$\overline{TFPR_Q} = \left(\int \frac{1}{TFPR_{Q,i}} \frac{p\left(x_i\right) x_i}{p_q Q} di \right)^{-1}.$$

In the non-distorted economy, without capital wedges, the level of TFP in the Q sector is

$$TFP_Q^{\star} = \left(\int_i \left(z_i \right)^{\frac{\mu}{1-\mu}} di \right)^{\frac{1-\mu}{\mu}} \equiv \overline{z}.$$

Therefore, we can measure the improvement in TFP in the Q sector, Ω_Q, as a result of eliminating the capital wedges, or equivalently, as a result of eliminating the collateral constraints:

$$\Omega_Q \equiv 1 - \frac{TFP_Q}{TFP_Q^{\star}} = 1 - \left(\int_i \left(\frac{\overline{z}}{z_i} \frac{TFPR_{Q,i}}{\overline{TFPR_Q}} \right)^{\frac{\mu}{1-\mu}} di \right)^{\frac{\mu-1}{\mu}}.$$

This measure does not capture the aggregate effect on the economy because (i) it applies only to the Q sector and not to the production of the final good, and (ii) it does not take into account changes in aggregate capital in the efficient economy with respect to the equilibrium of the distorted economy. In the benchmark calibration of Guvenen et al. (2023b) this variable has value of $\Omega_Q = 0.35$, implying TFP gains of 35% in the Q sector coming from eliminating the collateral constraints.

***Aggregate TFP.*.** The final goods producers operate competitively and face no constraints or distortions, so there is no labor misallocation in the model. Because of this, the only source of misallocation and TFP losses is the Q sector. We can therefore write output as

$$Y = \text{TFP} \cdot K^{\alpha} L^{1-\alpha},$$

where $\text{TFP} \equiv \text{TFP}_Q^{\alpha}$ captures the aggregate TFP of the model. Similarly, we can define the efficient TFP level of the economy as $\text{TFP}^{\star} \equiv (\text{TFP}^{\star})^{\alpha}$ and the aggregate TFP gain from

eliminating distortions in the economy as

$$\Omega_Y \equiv 1 - \frac{TFP}{TFP^\star} = 1 - \left(\frac{TFP_Q}{TFP_Q^\star} \right)^\alpha . \tag{17.1}$$

In the benchmark calibration, the total productivity gain from eliminating the collateral constraints in the Q sector amounts to 16% higher TFP.

Part V

Stochastic Calculus

This part of the course develops the mathematical tools necessary to study how random variables affect optimization problems. We start with an overview of the most relevant stochastic processes that we will encounter in the applications to come. The most important result is Ito's Lemma, which defines the way in which we can take derivatives of functions that depend on diffusions. Then we can apply Ito's Lemma to problems of dynamic optimization, with special attention to stopping time problems. Finally we apply it to the characterization of the distribution of a random variable. This is done by means of the Kolmogorov forward equation.

All these sections follow closely Dixit and Pindyck (1994), with some portions adapted from Stokey (2009).

18. Stochastic Processes

18.1. Definitions

The idea now is to study sequences of random variables. A stochastic process is similar to a random variable, with the difference that it also depends on time. Adding the time dimension adds notation, but it does not change any of the main ideas. For convenience we will first go over the definition of a random variable.

Definition 18.1. (Random variable) Let $(\Omega, \mathcal{A}, P)$ be a probability space and $x : \Omega \to \mathbb{R}$ a real valued function. x is a random variable if and only if x is measurable, that is, if and only if $x^{-1}(B) \in \mathcal{A}$ for all $B \in \mathcal{B}$, where $\mathcal{B}$ is the Borel σ-algebra on $\mathbb{R}$. We further establish the same notation:

(a) An outcome is an element $\omega \in \Omega$.

(b) An event is a measurable subset of Ω: $A \in \mathcal{A}$.

(c) The real number $x(\omega)$ is a realization of the random variable.

(d) The probability measure for x is then: $\mu(B) = P\left(x^{-1}(B)\right) = P\left(\left\{\omega \in \Omega | x(\omega) \in B\right\}\right)$, for $B \in \mathcal{B}$.

(e) The distribution function for f is: $G(b) = \mu\left((-\infty, b]\right)$, for $b \in \mathbb{R}$.

Now we can work on adding the time dimension to the definition of a random variable. In general time can be discrete or continuous, but in what follows we will assume that time is continuous starting at 0 and going on forever, so $t \in [0, \infty)$. Intuitively a stochastic process is formed by function $x : [0, \infty) \times \Omega \to \mathbb{R}$ that gives a realization for every outcome and time. At every point in time the random variable takes a variable, the sequence of those values forms the realization (path) of the stochastic process.

The question is on how to measure the possible outcomes of the random variable through time. We need a way of determining where the random variable is at a certain point in time, and where it has been, but that does not provide information about the value of future realizations. This is achieved using a filtration.

Definition 18.2. (Filtration) Let $\mathcal{A}$ be a σ-algebra. The set $\mathbb{A} = \left\{\mathcal{A}_t | t \geq 0\right\}$ is a filtration if $\mathcal{A}_t \subseteq \mathcal{A}$ and $\mathcal{A}_s \subseteq \mathcal{A}_t$ for all $t \geq 0$ and $s \leq t$. $\mathcal{A}_t$ is the set of events known at time t.

Now we can define a stochastic process as a function that is measurable in a filtered space.

Definition 18.3. (Stochastic Process) Let $(\Omega, \mathbb{A}, P)$ be a filtered probability space with a time index $t \in \mathbb{R}_+$, and let $\mathcal{B}_+$ be the Borel sets of $\mathbb{R}_+$. A stochastic process is a function $x : [0, \infty) \times \Omega \to \mathbb{R}$ that is measurable with respect to $\mathcal{B}_+ \times \mathcal{A}$ (that is, x is jointly measurable in (t, ω)). Moreover:

(a) For all $t \in \mathbb{R}_+$ and $\omega \in \Omega$, $x(t, \omega)$ is measurable with respect to $\mathcal{A}_t$, where $\mathcal{A}_t$ is in the filtration $\mathbb{A}$.

(b) For all $t \in \mathbb{R}_+$, $x(t, \cdot) : \Omega \to \mathbb{R}$ is an ordinary random variable on the probability space $(\Omega, \mathcal{A}_t, P_t)$.

(c) For all $\omega \in \Omega$, $x(\cdot, \omega) : \mathbb{R}_+ \to \mathbb{R}$ is a Borel measurable function. This is called the **sample path** of x.

18.2. Discrete time examples

It is not hard to come up with examples of discrete time stochastic processes. They are often used to model the behavior of many stationary economic variables by means of ARMA(p,q) representations, as well non-stationary variables usually related to random walks.

To fix ideas we start with the simple example of a (fair) coin toss. There are two possible outcomes, so $\Omega = \{H, T\}$, when tossing the coin is always possible to know which outcome occurred, and whether or not the coin was tossed, this gives: $\mathcal{A} = \left\{\{H\}, \{T\}, \emptyset, \Omega\right\}$. Finally the probability distribution P assigns values to sets in the σ-algebra $\mathcal{A}$:

$$P(\{H\}) = P(\{T\}) = \frac{1}{2} \qquad P(\emptyset) = 0 \qquad P(\Omega) = 1$$

Now we can define a random variable $\epsilon : \Omega \to \mathbb{R}$ as: $\epsilon(H) = 1$ and $\epsilon(T) = -1$. ϵ is a random variable with respect to the probability space $(\Omega, \mathcal{A}, P)$. As will be the case almost always we can dispense of the outcome space Ω for most applications and just refer to the random variable and the probability distribution induced over its values. In this way we have: $\epsilon \in \{-1, 1\}$ with $\Pr(\epsilon = 1) = \Pr(\epsilon = -1) = 1/2$.

Furthermore we can extend this example to define the stochastic process that comes up from the repeated coin toss. In this case time is discrete and finite $t \in \{1, 2, 3\}$ and at each time a coin is tossed, then the random variable variable ϵ_t is defined as the value of ϵ given the outcome of the t^{th} coin toss. The sequence $\{\epsilon_t\}_{t=1}^{3}$ is a stochastic process with respect to the filtered probability space $(\Omega, \mathbb{A}, P)$, where:

$$\Omega = \{(H, H, H), (H, H, T), (H, T, H), (H, T, T), (T, H, H), (T, T, H), (T, H, T), (T, T, T)\}$$

$$\mathcal{A} = 2^{\Omega} \qquad P(\omega) = \frac{1}{8} \quad \forall_{\omega \in \Omega}$$

The filtration is established taking into account that at each point in time only the outcome of current and past tosses is known:

$$\mathcal{A}_1 = \left\{\emptyset, \Omega, \{(H,H,H),(H,H,T),(H,T,H),(H,T,T)\}, \{(T,H,H),(T,T,H),(T,H,T),(T,T,T)\}\right\}$$

$$\mathcal{A}_2 = \left\{\emptyset, \Omega, \{(H,H,H),(H,H,T)\}, \{(T,H,H),(T,H,T)\}, \{(H,T,H),(H,T,T)\}, \{(T,T,H),(T,T,T)\}\right\}$$

$$\mathcal{A}_3 = \mathcal{A}$$

So, in the first σ-algebra all outcomes for which the first toss comes up heads are indistinguishable from each other, in the second σ-algebra one can distinguish between outcomes that have the sequence $\{H, T\}$ and $\{H, H\}$, but no information is given about the outcome of third toss. This same ideas apply if time goes on forever, so we can define our stochastic process over $t \in \mathbb{N}$.

In the previous example the stochastic process obtained satisfies the property of being iid (identically and independently distributed). The values of the stochastic process at each point in time are independent from its previous values, and they all have the same probabilities of occurring.

We now use our stochastic process $\{\epsilon_t\}$ to define a random walk. Random walks are particularly useful to understand the behavior of continuous time stochastic processes. As we will see the building block of most of them is the continuous time approximation of a random walk.

Example 18.1. (Random Walk Process) Consider a stochastic process x. Denote by x_t the value of x at time t, and fix the initial value x_0. x_t is assumed to evolve according to:

$$x_t = x_{t-1} + \epsilon_t \qquad \text{for } t \geq 1$$

ϵ_t is a random variable that can take two values $\{-1, 1\}$, and its probability distribution is independent of time, so that: $\Pr(\epsilon_t = 1) = \Pr(\epsilon_t = -1) = \frac{1}{2}$.

Given the starting value x_0 the variable x_t can only take on discrete values. For instance, for $x_0 = 0$ and t odd they are $\{-t, \ldots, -1, 0, 1, \ldots, t\}$, and for t even they are $\{-t, \ldots, -2, 0, 2, \ldots, t\}$. These values tell you which paths of the process cam be known at time t.

Finally, this process has no drift. Given an initial value x_0 the expected value of x_t

for any t is x_0 ($E[x_t] = x_0$), this follows from the expected value of each change being $E[x_t - x_{t-1}] = E[\epsilon_t] = 0$.

This process can be generalized in many ways. The most useful one for our purposes is to allow for drift, which can be done by changing the probabilities of the random variable ϵ_t, letting $\Pr(\epsilon_t = 1) = p$ and $\Pr(\epsilon_t = -1) = 1 - p$ achieves the desired result. If $p > 1/2$ the process will have positive drift.

18.3. Brownian motion (Wiener processes)

A Brownian motion, or Weiner process, is a continuous time stochastic process ($W(t)$) that satisfies three properties:

(a) $W(t)$ has continuous sample paths.

(b) $W(t)$ has stationary independent increments.

(c) Increments of $W(t)$ over a finite interval of time are normally distributed with variance that increases linearly in time.

The first property implies that a Brownian motion has no jumps, so as the time interval goes to zero the change in the process must also go to zero. The second and third properties imply that the change in $W(t)$ over some interval of length Δt must satisfy:

$$\Delta W = \epsilon_t \sqrt{\Delta t} \qquad \epsilon_t \sim N(0, 1)$$

which we write as $dW = \epsilon_t \sqrt{dt}$ as $\Delta t \to 0$. This implies that:

$$E[dW] = E[\epsilon_t]\sqrt{dt} = 0 \qquad V[dW] = E\left[\epsilon_t^2\right] dt = dt$$

Moreover we assume that ϵ_t is serially uncorrelated, i.e., $E[\epsilon_t \epsilon_s] = 0$ for $t \neq s$, so the values of dW for any two different time intervals are independent.

Its easy to note the relation between the Brownian motion and the random walk processes. In discrete time we had $\Delta x_t = x_t - x_{t-1} = \epsilon_t \Delta t$, where $\Delta t = 1$. We will use this fact when approximating Brownian motions using random walks as $\Delta t \to 0$.

To see that this representation implies the third property consider a time interval that starts at t and ends at T, and divide into n intervals of length $\Delta t = T/n$. Then we have:

$$W(t + T) - W(t) = \sum_{i=1}^{n} \epsilon_i \sqrt{\Delta t}$$

What we want to show is that $W(t+T) - W(t) \sim N(0,t)$. To prove this we can use the Central Limit Theorem:

Theorem 18.1. ***(Central Limit Theorem)*** *If* $\{\epsilon_1, \epsilon_2, \epsilon_3, \ldots\}$ *are iid (but not necessarily normal) with* $E\left[\epsilon_i\right] = \mu < \infty$ *and* $V\left[\epsilon_i\right] = \sigma^2 < \infty$*, then* $Z_n = \sqrt{n}\frac{\sum_{i=1}^{n}\epsilon_i - n\mu}{\sigma} \to N(0,1)$ *as* $n \to \infty$.

Note that ϵ_i already satisfies being iid and $E\left[\epsilon_i\right] = 0$ and $V\left[\epsilon_i\right] = 1$, so $Z_n = \sqrt{n}\sum_{i=1}^{n}\epsilon_i$. Then we can write:

$$W(t+T) - W(t) = \sqrt{T}Z_n$$

By the CLT this converges to a $N(0,T)$ as $n \to \infty$.

A Brownian motion can be generalized to have drift μ and variance σ^2. This is done by adjusting the way the increments of the stochastic process work:

$$dx = \mu dt + \sigma dW$$

In this case the increments are given by a non-stochastic component μdt, which indicates that the process will drift by μ per unit of time deterministically if there are no shocks, and by a stochastic component σdW, where σ is scaling the variance of the increments of the Weiner process W. This process satisfies:

$$E\left[dx\right] = \mu dt \qquad V\left[dx\right] = \sigma^2 dt$$

18.3.1. Random walk approximation of a Brownian motion

As mentioned above we can use the similarities between the increments of a Brownian motion and the increments of a random walk to approximate continuous time processes using discrete time ones. This is important because of two reasons: it helps explain the mechanics of the continuous time model, and it provides an algorithm for simulation in the computer.

Our objective is to approximate the a Brownian motion with drift:

$$dx = \mu dt + \sigma dW$$

We will approximate with a discrete time process y whose increments are h with probability p and $-h$ with probability $1-p$. This gives:

$$E\left[\Delta y\right] = ph - (1-p)h = (2p-1)h$$

$$V\left[\Delta y\right] = E\left[(\Delta y)^2\right] - \left(E\left[\Delta y\right]\right)^2 = \left(1 - (2p-1)^2\right) h^2$$

In order to get the approximation we need to choose values for $h,\ p$ and Δt so that:

$$\mu\Delta t = (2p-1)\, h$$
$$\sigma^2\Delta t = 4p\,(1-p)\, h^2$$

Solving for p we get:

$$p^2 - p + \frac{\sigma^2}{4\left(\sigma^2 + \mu^2\Delta t\right)} = 0$$

The roots of these equation are:

$$\begin{aligned} p &= \frac{1}{2}\left(1 \pm \sqrt{1 - \frac{\sigma^2}{\left(\sigma^2 + \mu^2\Delta t\right)}}\right) \\ &= \frac{1}{2}\left(1 \pm \frac{\mu\sqrt{\Delta t}}{\sqrt{\sigma^2 + \mu^2\Delta t}}\right) \\ &\approx \frac{1}{2}\left(1 \pm \frac{\mu}{\sigma}\sqrt{\Delta t}\right) \end{aligned}$$

where the approximation follows if Δt is small enough relative to σ^2/μ^2, because we are taking Δt close to zero this assumption is satisfied. We further choose only the "+" root because that way $p \geq 1/2$ when $\mu \geq 0$.

Now we can find a value for h:

$$\begin{aligned} \sigma^2\Delta t &= 4p\,(1-p)\, h^2 \\ \sigma^2\Delta t &= 2\left(1 + \frac{\mu}{\sigma}\sqrt{\Delta t}\right)\left(1 - \frac{1}{2}\left(1 + \frac{\mu}{\sigma}\sqrt{\Delta t}\right)\right) h^2 \\ \sigma^2\Delta t &= \left(1 + \frac{\mu}{\sigma}\sqrt{\Delta t}\right)\left(1 - \frac{\mu}{\sigma}\sqrt{\Delta t}\right) h^2 \\ \sigma^2\Delta t &= \left(1 - \left(\frac{\mu}{\sigma}\right)^2 \Delta t\right) h^2 \\ \sigma^2\Delta t &\approx h^2 \\ \sigma\sqrt{\Delta t} &\approx h \end{aligned}$$

As before we can disregard the term $\left(\frac{\mu}{\sigma}\right)^2 \Delta t$ as long as Δt is small enough relative to σ^2/μ^2.

As an exercise can verify that the first equation also holds:

$$\mu\Delta t = (2p-1)\, h$$

$$\mu\Delta t = (2p-1)\,\sigma\sqrt{\Delta t}$$
$$\frac{\mu}{\sigma}\sqrt{\Delta t} = 2p-1$$
$$\frac{1}{2}\left(1+\frac{\mu}{\sigma}\sqrt{\Delta t}\right) = p$$

In order to simulate a Brownian motion with parameters (μ, σ) we can do as follows:

(a) Set a Δt small relative to $\frac{\sigma^2}{\mu^2}$.

(b) Set $p = \frac{1}{2}\left(1+\frac{\mu}{\sigma}\sqrt{\Delta t}\right)$ and $h = \sigma\sqrt{\Delta t}$.

(c) Simulate the increments of x by drawing realization of a random variable ϵ_t that takes value h with probability p and $-h$ with probability $1-p$.

18.4. Ito processes

Ito processes are the generalization of Brownian motions. Their drift and variance is allowed to depend on the level of the process and the time:

$$dx = \mu(x,t)\,dt + \sigma(x,t)\,dW \tag{18.1}$$

where the functions μ and σ give the value of the mean and standard deviations of the increments of the process x:

$$\mu(x,t) = \lim_{\Delta\to 0^+}\frac{1}{\Delta}E\left[x(t+\Delta)-x(t)\,|x(t)=x\right]$$
$$(\sigma(x,t))^2 = \lim_{\Delta\to 0^+}\frac{1}{\Delta}E\left[(x(t+\Delta)-x(t))^2\,|x(t)=x\right]$$

For future reference note that an Ito process can also be represented as:

$$x(t) = x(0) + \int_0^t \mu(x,s)\,ds + \int_0^t \sigma(x,s)\,dW(s)$$

where the last term is a stochastic integral. Stochastic integrals play an important role in the theory of stochastic processes, for now it suffices to state the following result.

Proposition 18.1. *Let $x(t)$ be an integrable function, then $E\left[\int_0^t x(s)\,dW(s)\right] = 0$.*

This proposition states that the expected value of a stochastic integral is identically zero. The derivation of the result, along with other properties can be found in Stokey (2009, Sec. 3.2).

Two Ito process are of particular importance. They are presented in the examples below.

Example 18.2. (Geometric Brownian notion) A Geometric Brownian motion is an Ito process with $\mu(x, t) = \mu x$ and $\sigma(x, t) = \sigma x$, so:

$$dx = \mu x dt + \sigma x dW$$

A geometric Brownian motion can be thought of as a Brownian motion where the properties apply to percentage increments instead of increments:

$$\frac{dx}{x} = \mu dt + \sigma dW$$

So the percentage increment, dx/x, are normally distributed with mean $\mu\Delta t$ and variance $\sigma^2\Delta t$.

Example 18.3. (Ornstein-Uhlenbeck process) Unlike the previous processes an OU process is mean reverting, similar to an AR(1) process in discrete time. An OU process is an Ito process with $\mu(x, t) = \mu(\overline{x} - x)$ and $\sigma(x, t) = \sigma$. If $x > \overline{x}$ then the process drifts down, and if $x < \overline{x}$ the process drifts up.

$$dx = \mu(\overline{x} - x)\, dt + \sigma dW$$

18.5. Jump processes - Poisson Processes

Jump processes are a type of stochastic process that has discontinuous paths. Jump process change by discrete amounts when a certain outcome occurs. The most important Jump process is the Poisson process, which is just a jump process such that the time of the jumps follows a Poisson distribution. To define it let λ be the mean arrival rate of a jump and u the size of the change of the process (usually $u = 1$, but in general u can be itself a random variable). Then for some process q we have:

$$dq = \begin{cases} 0 & \text{with prob. } 1 - \lambda dt \\ u & \text{with prob. } \lambda dt \end{cases}$$

We can now define a more general process that depends on the Jump process q:

$$dx = f(x, t)\, dt + g(x, t)\, dq \tag{18.2}$$

where absent a jump x evolves deterministically according to the function f, and when there is a jump it moves according to function g,

$$E\left[dx\right] = f(x, t)\, dt + \lambda E_u\left[g(x, t)\, u\right] dt.$$

19. Ito's Lemma

We are often concerned with the behavior of functions of stochastic processes, in particular the differentials of those functions. The number one example at hand is to know how the value of an asset (or an option) evolves over time. Ito's Lemma gives a way to compute those differentials. This relates the functions we are interested in to the stochastic differential equation that governs the underlying stochastic process.

Consider a function $F(x, t)$ that depends on a stochastic process x. x is assumed to be an Ito process following:

$$dx = \mu(x, t)\, dt + \sigma(x, t)\, dW \tag{19.1}$$

Normal calculus rules would give the differential of F as:

$$dF = \frac{\partial F}{\partial x} dx + \frac{\partial F}{\partial t} dt$$

Although not always clear, one of the reasons for expressing the differential without resorting to higher order terms is that those terms depend on dt^2, $dt^3 \ldots$. As $dt \to 0$ all higher order terms go to zero faster, and are hence ignored. But stochastic process add a new factor because their components depend of time through $\sqrt{dt}$, so square terms like $(dx)^2$ must also be considered.

A second order Taylor expansion of F gives:

$$dF = \frac{\partial F}{\partial x} dx + \frac{\partial F}{\partial t} dt + \frac{1}{2}\left(\frac{\partial^2 F}{\partial x^2}(dx)^2 + \frac{\partial^2 F}{\partial t^2}(dt)^2 + \frac{\partial^2 F}{\partial t^2}(dx)(dt)\right)$$

As shown in Øksendal (2003, Sec. 4.1) $dWdt = dt^2 = 0$, they can be safely ignored because they depend on terms of order higher than dt. That leaves us with:

$$dF = \frac{\partial F}{\partial x} dx + \frac{\partial F}{\partial t} dt + \frac{1}{2}\frac{\partial^2 F}{\partial x^2}(dx)^2 \tag{19.2}$$

From the definition of our Ito process we get:

$$\begin{aligned}(dx)^2 &= \left(\mu^2(x, t)(dt)^2 + 2\mu(x, t)\,\sigma(x, t)\, dtdW + \sigma^2(x, t)(dW)^2\right) \\ &= \sigma^2(x, t)(dW)^2\end{aligned}$$

We can again drop the terms involving $(dt)^2$ and $(dtdW)$, and also show that $(dW)^2 = dt$ (recall that $E\left[(dW)^2\right] = dt$). The proof is not hard and can be found in Øksendal (2003, Sec.

4.1). Replacing:

$$dF = \left(\frac{\partial F}{\partial x}dx + \frac{\partial F}{\partial t}dt\right) + \frac{1}{2}\frac{\partial^2 F}{\partial x^2}\left(\sigma^2(x,t)\,dt\right)$$
$$dF = \left(\frac{\partial F}{\partial t} + \mu(x,t)\frac{\partial F}{\partial x} + \frac{1}{2}\sigma^2(x,t)\frac{\partial^2 F}{\partial x^2}\right)dt + \sigma(x,t)\frac{\partial F}{\partial x}dW \tag{19.3}$$

This derivation (Ito's Formula) means that $y = F(x,t)$ is itself an Ito process with $\mu_y(x,t) = \left(F_t + \mu(x,t)F_x + \frac{1}{2}\sigma^2(x,t)F_{xx}\right)$ and $\sigma_y(x,t) = \sigma(x,t)F_x$ as parameters. Unsurprisingly, the expected value of y is $\mu_y(x,t)$ and its variance is $\sigma_y(x,t)\,dt$.

19.1. Application to geometric brownian motion

We can use Ito's Lemma to obtain the properties of different stochastic processes. For instance the Geometric Brownian motion can be shown to be the exponential of a standard brownian motion, or equivalently it can be shown that the logarithm of a geometric brownian motion is a brownian motion.

Let x be a geometric brownian motion satisfying:

$$dx = \mu x dt + \sigma x dW$$

and $y = \ln x$. By Ito's Lemma:

$$\begin{aligned} dy &= \left(\mu x \cdot \frac{1}{x} + \frac{1}{2}\sigma^2 x^2 \cdot \left(\frac{-1}{x^2}\right)\right)dt + \sigma x \cdot \frac{1}{x}dW \\ &= \left(\mu - \frac{1}{2}\sigma^2\right)dt + \sigma dW \end{aligned}$$

thus y is a brownian motion with parameters $\mu_y = \mu - \frac{1}{2}\sigma^2$ and $\sigma_y = \sigma$. The drift of y is lower than the drift of x, because the logarithm is a concave function Jensen's inequality implies that the expected value of the log is lower.

We can also obtain the expected value of x by noting that:

$$x(t) = x(0) + \int_0^t \mu x(s)\,ds + \int_0^t \sigma x(s)\,dW(s)$$

taking expectations gives:

$$E[x(t)] = x(0) + \int_0^t \mu E[x(s)]\,ds$$

recalling that the third term is a stochastic integral, and hence has expected value equal to zero. From this equation we can derive a first order differential equation for the expected value of x:

$$dE[x] = \mu E[x]\, dx$$

The solution for this equation, given the boundary condition $E[x(0)] = x(0)$ is:

$$E[x] = x(0)\, e^{\mu t}$$

Finding the variance (and other moments) works in the same way. For the variance we want to obtain an expression for x^2, so first consider the function $f(x) = x^2$. By Ito's Lemma we get:

$$df = \left(2\mu x^2 + \sigma^2 x^2\right) dt + 2\sigma x^2 dW$$
$$x^2 = f(x) = x_0^2 + \left(2\mu + \sigma^2\right) \int_0^t x^2(s)\, ds + 2\sigma \int_0^t x^2(s)\, dW(s)$$

We can now take expectations to obtain:

$$E\left[x^2\right] = x_0^2 + \left(2\mu + \sigma^2\right) \int_0^t E\left[x^2(s)\right] ds$$

which leads to a differential equation for $E\left[x^2\right]$:

$$dE\left[x^2\right] = \left(2\mu + \sigma^2\right) E\left[x^2\right]$$
$$E\left[x^2\right] = x^2(0)\, e^{\left(2\mu+\sigma^2\right)t}$$

the variance is then:

$$\begin{aligned} V[x] &= E\left[x^2\right] - E[x]^2 \\ &= x^2(0)\, e^{\left(2\mu+\sigma^2\right)t} - x^2(0)\, e^{2\mu t} \\ &= x^2(0)\, e^{2\mu t}\left(e^{\sigma^2 t} - 1\right) \end{aligned}$$

Some applications are shown below:

Example 19.1. Consider an asset that gives flow payoffs x that evolve according to a geometric brownian motion

$$dx = \mu x dt + \sigma x dW$$

we can compute the expected discounted value of holding that asset easily using the results above:

$$E\left[\int_0^\infty e^{-\rho t}x(t)\,dt\right] = \int_0^\infty e^{-\rho t}E[x(t)]\,dt = \int_0^\infty x(0)\,e^{-(\rho-\mu)t}dt = \frac{x_0}{\rho-\mu}$$

Example 19.2. Now consider an agent that receives flow consumption of x, which evolves again as a geometric brownian motion. The agent's utility is CRRA, so that $u(x) = \frac{x^{1-\theta}}{1-\theta}$. We want to know the expected present value of utility.

$$E\left[\int e^{-\rho t}u(x(t))\,dt\right] = \int e^{-\rho t}E[u(x(t))]\,dt$$

To know it we need to compute $E[u(x(t))]$. From Ito's Lemma we have:

$$\begin{aligned}
du &= \left(\mu x\cdot x^{-\theta} + \frac{1}{2}\sigma^2x^2\cdot\left(-\theta x^{-\theta-1}\right)\right)dt + \sigma x\cdot x^{-\theta}dW\\
du &= (1-\theta)\left(\mu - \frac{\theta}{2}\sigma^2\right)\frac{x^{1-\theta}}{1-\theta}dt + (1-\theta)\,\sigma\frac{x^{1-\theta}}{1-\theta}dW\\
du &= (1-\theta)\left(\mu - \frac{\theta}{2}\sigma^2\right)udt + (1-\theta)\,\sigma u dW
\end{aligned}$$

Thus, u is itself a geometric brownian motion (actually if x is a brownian motion x^k is a geometric brownian motion). Using our previous results we have:

$$E[u] = u(x(0))\,e^{(1-\theta)\left(\mu-\frac{\theta}{2}\sigma^2\right)t}$$

So we have:

$$E\left[\int e^{-\rho t}u(x(t))\,dt\right] = \int e^{-\rho t}E[u(x(t))]\,dt = \frac{(x(0))^{1-\theta}}{(1-\theta)\left(\rho-(1-\theta)\left(\mu-\frac{\theta}{2}\sigma^2\right)\right)}$$

19.2. Poisson Processes

Similar, and simpler, results can be obtained if x follows a Poisson process:

$$dx = f(x,t)\,dt + g(x,t)\,dq$$

and we have a function $H(x,t)$ that depends on x. Unlike the Ito Process the Poisson process does not depend on $\sqrt{dt}$, so higher order terms in the Taylor expansion can be ignored

altogether to get:

$$\begin{aligned} dH &= H_t dt + H_x dx \\ &= \left(H_t + f\left(x,t\right) H_x\right) dt + g\left(x,t\right) H_x dq \end{aligned}$$

The expected value of this change must take into account the probability of a jump in q (given by λdt), so we have:

$$E\left[dH\right] = \left(H_t + f\left(x,t\right) H_x\right) dt + \lambda E_u\left[H\left(x + ug\left(x,t\right), t\right) - H\left(x,t\right)\right] dt \tag{19.4}$$

it follows, by using the identity function that $E\left[dx\right] = f\left(x,t\right) dt + \lambda E_u\left[ug\left(x,t\right)\right] dt$.

We can apply this result to a couple examples taken from Dixit and Pindyck (1994):

Example 19.3. Consider an individual that lives forever and receives a wage $w\left(t\right)$ at each point in time. The wage increases by ϵ at random times, following a Poisson process with arrival rate λ, so:

$$dw = \epsilon dq$$

The individual wants to know the expected discounted value of taking the job we need to compute:

$$V\left(w\right) = E\left[\int_0^\infty e^{-\rho t} w\left(t\right) dt\right]$$

The function V (a value function) has a recursive representation, this is easier to see in the discrete time approximation. Consider a period of length Δt, then:

$$\begin{aligned} V\left(w\left(t\right)\right) &= w\left(t\right)\Delta t + \frac{1}{1+\rho\Delta t} E\left[V\left(w\left(t+\Delta t\right)\right)\right] \\ \left(1+\rho\Delta t\right) V\left(w\left(t\right)\right) &= \left(1+\rho\Delta t\right) w\left(t\right)\Delta t + E\left[V\left(w\left(t+\Delta t\right)\right)\right] \\ \rho\left(\Delta t\right) V\left(w\left(t\right)\right) &= \left(1+\rho\Delta t\right) w\left(t\right)\Delta t + E\left[V\left(w\left(t+\Delta t\right)\right) - V\left(w\left(t\right)\right)\right] \\ \rho V\left(w\left(t\right)\right) &= \left(1+\rho\Delta t\right) w\left(t\right) + \frac{E\left[\Delta V\right]}{\Delta t} \end{aligned}$$

Taking the limit as $\Delta t \to 0$ we get:

$$\rho V = w + \frac{E\left[dV\right]}{dt} \tag{19.5}$$

Staying in the job works just like an asset, with a normal return at rate ρ being equal to the sum of the dividend (in this case given by the wage) and the expected capital gains (from changes in the wage). In the expression above $\frac{E[dV]}{dt} = \lim_{\Delta t \to 0} \frac{1}{\Delta t} E\left[\Delta V\right]$

We can apply the formula from above:

$$E\left[dH\right] = \left(H_t + f\left(x,t\right)H_x\right)dt + \lambda E_u\left[H\left(x + ug\left(x,t\right),t\right) - H\left(x,t\right)\right]dt$$

where $H = V$, $x = w$, $f\left(x,t\right) = 0$ and $g\left(x,t\right) = \epsilon$ and $u = 1$ with certainty:

$$\begin{aligned} E\left[dV\right] &= \lambda\left(V\left(w+\epsilon\right) - V\left(w\right)\right)dt \\ &= \lambda\epsilon\left(\int_0^\infty e^{-\rho t}dt\right)dt \\ &= \frac{\lambda\epsilon}{\rho}dt \end{aligned}$$

This leaves us with an explicit solution for V:

$$V = \frac{w}{\rho} + \frac{\lambda\epsilon}{\rho^2}$$

V is equal to an asset that pays the current wage forever plus the capitalized value of the average raise in wages per unit of time.

Example 19.4. Consider now a firm that produces using capital. As long as capital is operational a flow profit of π is obtained, but capital becomes obsolete when new technologies arrive. These innovations occur at random times following a Poisson process with arrival rate λ. Once the innovation arrives and the capital becomes obsolete the firm goes out of business forever.

The value of the firm follows a process:

$$dV = -Vdq$$

The return can be found as before:

$$\rho V = \pi + \frac{1}{dt}E\left[dV\right]$$

To find $E\left[dV\right]$ we can again use our formula with $H = V$, the identity function:

$$E\left[dV\right] = -\lambda V dt$$

replacing we get

$$\rho V = \pi - \lambda V \quad \text{or} \quad V = \frac{\pi}{\rho + \lambda}.$$

Which this is equivalent to solving:

$$V = \int_0^\infty e^{-(\rho+\lambda)t}\pi dt$$

This should not be a surprise. Consider the case where there are no shocks and the firm can operate forever with certainty. Then V is:

$$V = \int_0^\infty e^{-\rho t}\pi dt = \frac{\pi}{\rho}$$

Now the firm shuts down with a certain probability, given by the arrival of the Poisson shock. Then:

$$V = E\left[\int_0^\infty e^{-\rho t}\pi dt\right] = \int_0^\infty \Pr\left[\text{No shock until time } t\right] e^{-\rho t}\pi dt$$

The probability of there being no shocks is known:

$$\Pr\left[\text{No shock until time } t\right] = e^{-\lambda t}$$

Replacing gives the desired result.

20. Dynamic Programming

In dynamic programming we aim to develop tools for solving problems that involve actions through time, that in turn affect the total value obtained by the agent takin the decisions. The key of dynamic programming is that it focuses on the current decision being taken and its effect on the continuation value for the agent, rather than try to solve for the whole sequence of actions at once.

20.1. Discrete time overview

Dynamic Programming

To build up to the concepts of dynamic programming in continuous time we will first consider a simple discrete time problem of a firm that must invest a fixed amount I to set up the operation of the firm. Once the firm is operational the firm produces one unit of good every period.The current price of the good is known and given by p_0, in the second period the price can go up or down:

$$p_1 = \begin{cases} (1+u)\ p_0 & \text{with prob. } q \\ (1-d)\ p_0 & \text{with prob. } 1-q \end{cases}$$

After that the price is constant. Hence, the firm's decision is whether to invest in the first period, in the second, or not to invest at all. It makes no sense to wait any longer because no new information will arrive after the initial change in price. The firm discounts future payments with an interest rate r.

We can solve the problem by tracing the decisions that the firm can take. In the second period, once the price is known and assuming that the firm is not yet in operation, the firm can either invest or not. If the firm does not invest it gets zero payoff, if it invests it gets:

$$F_1\left(p_1\right) = p_1 + \frac{p_1}{1+r} + \frac{p_1}{(1+r)^2} = p_1 \sum_{i=0}^{\infty} \frac{1}{(1+r)^i} = \frac{1+r}{r} p_1$$

The payoff of the firm is then:

$$V_1\left(p_1\right) = \max\left\{F_1\left(p_1\right) - 0, 0\right\}$$

Knowing this is relevant because if the firm does not invest in the first period it can always do so later, so V_1 constitutes the continuation payoff of the firm. The payoff to the firm if it

does not invest in the first period is then:

$$\frac{1}{1+r}E\left[V_1\left(p_1\right)\right] = \frac{1}{1+r}\left(qV_1\left(\left(1+u\right)\ p_0\right) + \left(1-q\right)V_1\left(\left(1-d\right)\ p_0\right)\right)$$

If the firm invests in the first period the payoff is:

$$\begin{aligned} F_0\left(p_0\right) &= p_0 + \frac{1}{1+r}E\left[F_1\left(p_1\right)\right] \\ &= p_0 + \frac{1}{1+r}\left(qF_1\left(\left(1+u\right)\ p_0\right) + \left(1-q\right)F_1\left(\left(1-d\right)\ p_0\right)\right) \\ &= p_0 + \left(\frac{q}{r}\left(1+u\right)\ p_0 + \frac{1-q}{r}\left(1-d\right)\ p_0\right) \\ &= \frac{1}{r}\left(1+r+q\left(u+d\right)-d\right)\ p_0 \end{aligned}$$

So, the value of the firm is:

$$V_0\left(p_0\right) = \max\left\{F_0\left(p_0\right) - I, \frac{1}{1+r}E\left[V_1\left(p_1\right)\right]\right\}$$

In this example we already see the basics of dynamics programming, splitting the problem into the decision at hand (invest or not invest) and the continuation value that they entail. The example also highlights one of the recurring topics of the course: option value. The firm has an option that allows it to invest any of the two dates. Waiting in this problem has value, because investing in the future also means to invest with better information. In fact we can compute the value of this option (to wait) by comparing the value that the firm would have if it was forced to take a decision in the first period:

$$\Omega_0\left(p_0\right) = \max\left\{F_0\left(p_0\right) - I, 0\right\}$$

with the value that includes the possibility of action in the second period:

$$V_0\left(p_0\right) - \Omega_0\left(p_0\right)$$

We can now extend this simple model to allow for action in many periods (more than two). Consider a firm that operates for $T < \infty$ periods. In each period the firm will choose the value of a control variable u that affects (potentially) the per-period payoffs of the firm, namely the profits, and the evolution of a random variable x. x is assumed to follow a Markov process so that the CDF of x_{t+1} is $\Phi_t\left(x_{t+1}|x_t, u_t\right)$. The random variable x is also allowed to affect payoffs, so per-period payoffs are: $\pi\left(u_t, x_t\right)$.

The firm discounts the future at rate $\frac{1}{1+\rho}$ and receives a final payoff pf $\Omega_T(x_T)$ in the last period. The objective is:

$$V_0(x_0) = \max_{\{u_t\}_{t=0}^{T-1}} E\left[\sum_{t=0}^{T-1}\left(\frac{1}{1+\rho}\right)^t \pi(u_t, x_t) + \left(\frac{1}{1+\rho}\right)^T \Omega_T(x_T)\right] \tag{20.1}$$

dynamic programming allows us to write the problem recursively. In the last period we have:

$$V_{T-1}(x_{T-1}) = \max_{u_{T-1}} \pi(u_{T-1}, x_{T-1}) + \left(\frac{1}{1+\rho}\right) E\left[\Omega_T(x_T) | x_{T-1}, u_{T-1}\right] \tag{20.2}$$

For all other periods we can use the notion of continuation payoffs to obtain:

$$V_t(x_t) = \max_{u_t} \pi(u_t, x_t) + \left(\frac{1}{1+\rho}\right) E\left[V_{t+1}(x_{t+1}) | x_t, u_t\right] \tag{20.3}$$

The problem can then be solved by backwards induction, choosing contingent plans for $u_t(x_t)$ one period at a time, instead of tackling the more complicated problem of choosing the whole sequence of $\{u_t\}$.

When time is not finite, there is no terminal date, and we cannot use backwards induction to solve the problem. In this case the value of the firm itself is also independent of time, because each period is just like the next. We then have:

$$V(x) = \max_{u} \pi(u, x) + \left(\frac{1}{1+\rho}\right) E\left[V\left(x'\right) | x, u\right] \tag{20.4}$$

the problem is now to find a function V that satisfies the equation above. The details behind the solution to this problem can be found in Stokey, Lucas, and Prescott (1989).

This setup is very versatile and can be applied to firm problems as the one above, but it is also at the core of modern macroeconomic theory. The following examples make this point in a non-stochastic version of the model.

Example 20.1. Consider an economy in which the representative consumer lives forever. There is a good in each period that can be consumed or saved as capital as well as labor. The consumer's utility function is

$$V\left(\bar{k}_0\right) = \sum_{t=0}^{\infty} \beta^t \log c_t$$

Here $0 < \beta < 1$. The consumer is endowed with 1 unit of labor in each period and with $\bar{k}_0$ units of capital in period 0. Capital fully depreciates each period. Feasible allocations

satisfy

$$c_t + k_{t+1} \leq \theta k_t^{\alpha} l_t^{1-\alpha}$$

Here $\theta > 0$ and $0 < \alpha < 1$. We can formulate the problem of maximizing the representative consumer's utility subject to feasibility conditions as a dynamic programming problem. The appropriate Bellman's equation is:

$$\begin{aligned} V(k) &= \max_{c,k',l} \left\{ \log c + \beta V\left(k'\right) \right\} \\ \text{s.t. } & c + k' \leq \theta k^{\alpha} l^{1-\alpha} \\ & c, k' \geq 0 \\ & 0 \leq l \leq 1 \end{aligned}$$

To solve it we guess that the value function has the form $a_0 + a_1 \log k$ and solve for the decisions of the consumer. The constraint will hold with equality because the utility function is strictly increasing in consumption, also production increases with labor and there is no disutility of it, hence there is a corner solution for labor indicating $l = 1$, so with the guess the problem becomes

$$a_0 + a_1 \log k = \max_{k' \in \left[0, \theta k^{\alpha} l^{1-\alpha}\right]} \log\left(\theta k^{\alpha} l^{1-\alpha} - k'\right) + \beta\left(a_0 + a_1 \log k'\right)$$

Then the FOC is

$$\frac{1}{\theta k^{\alpha} l^{1-\alpha} - k'} = \frac{\beta a_1}{k'}$$

solving for k'

$$\begin{aligned} k' &= \beta a_1 \left(\theta k^{\alpha} l^{1-\alpha} - k'\right) \\ &= \frac{\beta a_1 \left(\theta k^{\alpha} l^{1-\alpha}\right)}{1 + \beta a_1} \end{aligned}$$

Then plugging this back into the value function you get

$$a_0 + a_1 \log k = \log\left(\theta k^{\alpha} l^{1-\alpha} - \frac{\beta a_1 \left(\theta k^{\alpha} l^{1-\alpha}\right)}{1 + \beta a_1}\right) + \beta\left(a_0 + a_1 \log\left(\frac{\beta a_1 \left(\theta k^{\alpha} l^{1-\alpha}\right)}{1 + \beta a_1}\right)\right)$$

Collection terms with k you get

$$\begin{aligned} a_1 \log k &= \alpha \log k + \beta a_1 \alpha \log k \\ a_1 (\log k - \beta \alpha \log k) &= \alpha \log k \\ a_1 &= \frac{\alpha}{1 - \beta \alpha} \end{aligned}$$

which means the policy function is

$$\begin{aligned} k' &= \frac{\beta \frac{\alpha}{1-\beta\alpha} \left(\theta k^{\alpha} l^{1-\alpha} \right)}{1 + \beta \frac{\alpha}{1-\beta\alpha}} = \beta \alpha \theta k^{\alpha} l^{1-\alpha} \\ l &= 1 \\ c &= \theta k^{\alpha} l^{1-\alpha} - \beta \alpha \theta k^{\alpha} l^{1-\alpha} \end{aligned}$$

Optimal Stopping Time

There is another type of problem that deserves special treatment. Optimal stopping time problems are at the core of the continuous time applications in the rest of the course. In these problems the agent faces a binary choice (instead of a continuous choice as in the example above), they resemble the example of the firm at the beginning of the Section where the firm has to choose whether or not to invest. This problems are characterized by the inaction of the agent, because the agent usually acts just once, and most of the time the optimal choice is to do nothing. To characterize these problems let $\Omega(x)$ be the termination payoff received once the action is taken (and time is stopped). It depends on the value of state x. The Bellman equation is now:

$$V(x) = \max \left\{ \Omega(x), \max_u \pi(u, x) + \left(\frac{1}{1+\rho} \right) E \left[V\left(x' \right) | x, u \right] \right\} \tag{20.5}$$

We can now define a stopping time as a random variable that signals the decision to stop and take the termination payoff $\Omega(x)$. So:

$$T^{\star} = \left\{ x | \Omega(x) \geq \max_u \pi(u, x) + \left(\frac{1}{1+\rho} \right) E \left[V\left(x' \right) | x, u \right] \right\} \tag{20.6}$$

In general $T^{\star}$ can take many forms, but in most (if not all) of the relevant economic applications it will take the form: $T^{\star} = [\overline{x}, \infty)$, $T^{\star} = (-\infty, \underline{x}]$ or $T^{\star} = (-\infty, \underline{x}] \cup [\overline{x}, \infty)$. As an example we apply these ideas to the problem of search and unemployment, the McCall search model.

Example 20.2. Consider the following infinite horizon model. An agent searches for a job. Each period the agent receives a wage offer from a distribution $F(w)$ with bounded support $W = \left[0, \overline{W}\right]$. If accepted the agent will remain employed at that wage forever. If rejected the worker receives unemployment benefits b. Wage offers are iid over time. The worker preferences are $\sum \beta^t c_t$. Assume no borrowing or lending.

We first set up the workers decision as a dynamic programming problem:

$$V^E(w) = \frac{w}{1-\beta}$$
$$V^U = b + \beta \int \max\left\{V^E(\tilde{w}), V^U\right\} dF(\tilde{w})$$

The decision of a worker when facing a wage offer w is to accept it or reject it, the worker will accept if $V^E(w) > V^u$ and reject otherwise. Then the value of the worker is:

$$V(w) = \max\left[V^E(w), V^U\right]$$
$$V(w) = \max\left[\frac{w}{1-\beta}, b + \beta \int V(\tilde{w})\, dF(\tilde{w})\right]$$

Now we need to show that the decision to take action (accept a job offer) is given by $T^\star = [\overline{w}, \infty)$, where $\overline{w}$ is the reservation wage. To show this, note that V^U is independent of the wage and that V^E is increasing in wages. The reservation wage satisfies:

$$\frac{\overline{w}}{1-\beta} = b + \beta \int V(\tilde{w})\, dF(\tilde{w})$$

This implies that V is constant for $w < \overline{w}$, because the offers are rejected, and it is equal to V^E for $w \geq \overline{w}$:

$$V(w) = \begin{cases} \frac{\overline{w}}{1-\beta} & \text{if } w < \overline{w} \\ \frac{w}{1-\beta} & \text{if } w \geq \overline{w} \end{cases}$$

It is left to find $\overline{w}$. To do this we should first solve for V^U:

$$\begin{aligned} V^U &= b + \beta \int \max\left\{V^E(\tilde{w}), V^U\right\} dF(\tilde{w}) \\ &= b + \beta \int_0^{\overline{w}} \frac{\overline{w}}{1-\beta} dF(\tilde{w}) + \beta \int_{\overline{w}}^{\overline{W}} \frac{\tilde{w}}{1-\beta} dF(\tilde{w}) \\ &= b + \frac{\beta}{1-\beta}\left(\int_0^{\overline{w}} \overline{w} dF(\tilde{w}) + \int_{\overline{w}}^{\overline{W}} \tilde{w} dF(\tilde{w})\right) \end{aligned}$$

$$= b + \frac{\beta}{1-\beta}\left(\overline{w} - \int_{\overline{w}}^{\overline{W}} \overline{w} dF(\tilde{w}) + \int_{\overline{w}}^{\overline{W}} \tilde{w} dF(\tilde{w})\right)$$
$$= b + \frac{\beta}{1-\beta}\left(\overline{w} + \int_{\overline{w}}^{\overline{W}} (\tilde{w} - \overline{w}) dF(\tilde{w})\right)$$

The agent knows she is guaranteed to have $\overline{w}$ forever, finding a job just adds to the value with the wage in excess of $\overline{w}$.

Turning back to determining $\overline{w}$ we can replace V^U to get:

$$\overline{w} = b + \frac{\beta}{1-\beta} \int_{\overline{w}}^{\overline{W}} (\tilde{w} - \overline{w}) dF(\tilde{w})$$

This equation is guaranteed to have a solution for $\overline{w} \in \left[c, \overline{W}\right]$. The LHS is increasing in $\overline{w}$, while the RHS is decreasing in $\overline{w}$.

20.2. Continuous time dynamic programming

We can now turn to develop a general framework to solve dynamic problems in continuous time. To start consider the problem developed in the previous section with periods of length Δt. The agent receives a payoff $\pi(u, x)\Delta t$ every period (where π is the payoff flow), and discounts the future at a rate ρ per unit of time, so the effective discount rate for the period of length Δt is: $\frac{1}{1-\rho\Delta t}$. This leads to the following Bellman-type equation

$$V(x) = \max_u \pi(u, x)\Delta t + \left(\frac{1}{1+\rho\Delta t}\right) E\left[V\left(x'\right)|x, u\right] \tag{20.7}$$

Rearranging we get:

$$\rho V(x) = \max_u (1+\rho\Delta t)\pi(u, x) + \frac{E\left[\left(V\left(x'\right) - V(x)\right)|x, u\right]}{\Delta t} \tag{20.8}$$

Taking the limit as $\Delta t \to 0$ we get our continuous time Bellman equation:

$$\rho V(x) = \max_u \pi(u, x) + \frac{1}{dt} E\left[dV(x)|x, u\right] \tag{20.9}$$

where

$$\frac{E\left[dV\right]}{dt} = \lim_{\Delta t \to 0} \frac{1}{\Delta t} E\left[\Delta V\right]$$

Equation (20.9) works just like a non-arbitrage condition. We can thing of the agent as

holding an asset with value V. The LHS gives the normal rate of return per unit time that the agent requires to hold the asset, given the discount rate ρ. The RHS gives the effective payoff of the asset, composed by the immediate flow payoff π, and the expected capital gains (brought up by changes in the value of the asset).

We can further characterize the problem given knowledge of the stochastic process that x follows. This will allow us to evaluate the expectation in (20.9). If x follows an Ito process, as in equation (18.1), then Ito's Lemma gives the following result:

$$dV = \left(\mu(x,t)V' + \frac{1}{2}\sigma^2(x,t)V''\right)dt + \sigma(x,t)V'dW$$
$$E[dV] = \left(\mu(x,t)V' + \frac{1}{2}\sigma^2(x,t)V''\right)dt$$

Replacing we get the Hamilton-Jacobi-Bellman equation:

$$\rho V(x) = \max_u \pi(u,x) + \mu(x,t)V'(x) + \frac{1}{2}\sigma^2(x,t)V''(x) \tag{20.10}$$

We can take FOC with respect to u and then get a differential equation for V that we can solve.

If x follows a Poisson process, like the one in equation (18.2), we can obtain a similar result. From equation (19.4) we can compute $E[dV]$:

$$E[dV] = \left(f(x,t)V'(x)\right)dt + \lambda E_u\left[V(x+ug(x,t)) - V(x)\right]dt$$

Optimal Stopping Time and the Smooth Pasting Condition

We now go back to the stopping time problem reviewed in Section 20.1. Consider then the problem of an agent that is engaged in some activity (say running a firm). The agent gets a flow payoff of $\pi(x)$ if she continues with the activity, and $\Omega(x,t)$ if she quits the activity (stops). The value of the agent is:

$$V(x,t) = \max\left\{\Omega(x,t), \pi(x)\Delta t + \frac{1}{1+\rho\Delta t}E\left[V(x+dx, t+\Delta t)\right]\right\} \tag{20.11}$$

where x follows a diffusion process and in equation (18.1). We assume that Ω is continuous and weakly increasing in x.

In order to solve the problem we need to find regions of x where it is best for the agent

to continue and those for which it is best to stop. If x is in the continuation region then:

$$V(x) = \pi(x)\Delta t + \frac{1}{1+\rho\Delta t} E\left[V\left(x'\right)\right]$$

From above we know that this implies that for x in the continuation region we have (by applying Ito's lemma):

$$\rho V(x,t) = \pi(x) + V_t(x,t) + \mu(x,t) V_x(x,t) + \frac{1}{2}\sigma^2(x,t) V_{xx}(x,t)$$

For simplicity we assume now that the continuation region has the form $x \geq x^\star(t)$. It is only for $x \geq x^\star(t)$ that the equation above holds. In order to solve it we need to impose certain boundary conditions.

By assumption we know that $V(x) = \Omega(x,t)$ for $x < x^\star(t)$, then, by continuity we can impose that:

$$V\left(x^\star(t), t\right) = \Omega\left(x^\star(t), t\right)$$

this is called "value-matching". Continuity at $x^\star$ is actually necessary for a solution. Suppose for a contradiction that it is optimal to stop for $x < x^\star(t)$, but that $V(x^\star(t), t) < \Omega(x^\star(t), t)$, because V has to be continuous in the domain $x \geq x^\star(t)$ (because it is the solution to a differential equation), and Ω is continuous by definition, then it holds that for x to the right of $x^\star(t)$, but sufficiently close to $x^\star(t)$ it also holds that $V(x,t) < \Omega(x,t)$, which contradicts $x \geq x^\star(t)$ being the continuation region. A similar argument applies for the other inequality.

But this condition is not sufficient to solve the problem, because the value of $x^\star(t)$ is still unknown. The condition that allows us to solve the problem (of jointly finding V and $x^\star$) is to impose further smoothness to our value function, it must not only be continuous, but continuously differentiable. This condition is called "smooth pasting" and it requires the first derivative of the value function to be continuous, that is:

$$V_x\left(x^\star(t), t\right) = \Omega_x\left(x^\star(t), t\right)$$

The reason behind the smooth pasting condition is not at all evident. I will illustrate it below, but I recommend checking Appendix C of Chapter 4 in Dixit and Pindyck (1994), or Stokey (2009, Prop 6.4 pg 124).

To see why the smooth pasting condition arise consider the following case built for a contradiction: the value matching condition holds, but the smooth pasting condition fails, hence V and Ω must meet at a kink. There are two options:

(a) There is an upward kink (forming a concave function). If this is the case then, by continuity, Ω would be higher than F for some value $x > x^\star$. Contradicting that the continuation region starts at $x^\star$.

(b) There is a downward kink (forming a convex function). If this is the case then $x^\star$ cannot be a point of indifference either. There is a better strategy, namely continuing for some time Δt and then choosing what to do. This strategy give higher (expected) payoff.
To see this recall the random walk formulation of the brownian motion, in a time lapse Δt x can either go up by h with probability p or down by $-h$ with probability $1-p$, where:

$$h = \sigma\sqrt{\Delta t} \qquad p = \frac{1}{2}\left(1 + \frac{\mu}{\sigma}\sqrt{\Delta t}\right)$$

Then the agent can continue if the step is upward and stop if it is downward. The expected payoff of this strategy is:

$$V\left(x^\star(t), t\right) = \pi\left(x^\star(t), t\right)\Delta t + \frac{1}{1+\rho\Delta t}\left[pV\left(x^\star(t) + h, t + \Delta t\right) + (1-p)\,\Omega\left(x^\star(t) - h, t + \Delta t\right)\right]$$

We can take a Taylor expansion around $(x^\star(t), t)$ to approximate the value of $V(x^\star(t) + h, t + \Delta t)$ and $\Omega(x^\star(t) - h, t + \Delta t)$:

$$V\left(x^\star(t) + h, t + \Delta t\right) \approx V\left(x^\star(t), t\right) + V_x\left(x^\star(t), t\right)h + V_t\left(x^\star(t), t\right)\Delta t$$
$$\Omega\left(x^\star(t) - h, t + \Delta t\right) \approx \Omega\left(x^\star(t), t\right) - \Omega_x\left(x^\star(t), t\right)h + \Omega_t\left(x^\star(t), t\right)\Delta t$$

Replacing gives:

$$V\left(x^\star(t), t\right) = \pi\left(x^\star(t), t\right)\Delta t + \frac{1}{1+\rho\Delta t}\left(V\left(x^\star(t), t\right) + \frac{1}{2}\left[\left(V_x\left(x^\star(t), t\right) - \Omega_x\left(x^\star(t), t\right)\right)h + \left(V_t\left(x^\star(t)\right.\right.\right.\right.$$

where we use the value matching condition and the fact that $ph \approx \frac{1}{2}\sigma\sqrt{\Delta t}$ and $p\Delta t \approx \frac{1}{2}\Delta t$.
What matters for evaluating the strategy is the continuation value, and that Δt is of order h^2, so the first two terms in the continuation value $\left(V(x^\star(t), t) + \frac{1}{2}h\left(V_x(x^\star(t), t) - \Omega_x(x^\star(t), t)\right)\right)$ will dictate the behavior of the gain as $\Delta t \to 0$ (or equivalently $h \to 0$). These terms are positive as long as $V_x(x^\star(t), t) > \Omega_x(x^\star(t), t)$, which is the case if there is a downward kink.
Then there cannot be a downward kink, because it would contradict the optimality of the strategy of stopping at $x^\star(t)$.

Example 20.3. Consider a firm that has flow revenues of e^{x_t}, and that can be closed at any

time and sold for a value $\Omega > 0$. The owner of the firm is risk neutral and discounts the future at a rate $\rho > 0$. x_t follows:

$$dx_t = \mu dt + \sigma dW$$

The problem of the firm's owner is then:

$$V(x) = \max\left\{\Omega, e^x \Delta t + \frac{1}{1+\rho\Delta t} E\left[V(x+dx)\right]\right\}$$

where continuation is optimal for $x \geq x^\star$. This problem is independent of time.

As long as x is in the continuation region the value function satisfies the HJB equation:

$$\rho V(x) = e^x + \mu V_x(x) + \frac{1}{2}\sigma^2 V_{xx}(x)$$

This is a second order ordinary differential equation with constant coefficients. Then we know that the solution has the form:

$$V(x) = V^P(x) + A_1 H_1(x) + A_2 H_2(x)$$

where V^P is a particular solution to the differential equation, H_1 and H_2 are homogenous solutions, and $A_1$and A_2 are constants to be determined.

The particular solution is easy to obtain. We can solve for the value of never stopping:

$$V^P(x) = E\left[\int_0^\infty e^{-\rho t} e^x dt\right]$$

We can solve this expectation using the results in example 19.1. We get:

$$V^P(x) = \frac{x}{\rho - \left(\mu + \frac{1}{2}\sigma^2\right)}$$

we assume that $\rho - \left(\mu + \frac{1}{2}\sigma^2\right) > 0$ in order to guarantee the existence of this solution.

The homogenous solutions are obtained from the homogenous equation:

$$\rho H(x) = \mu H_x(x) + \frac{1}{2}\sigma^2 H_{xx}(x)$$

by guessing that $H(x) = e^{\xi x}$ and replacing we get:

$$\rho e^{\xi x} = \mu \xi e^{\xi x} + \frac{1}{2}\sigma^2 \xi^2 e^{\xi x}$$

$$0 = -\rho + \mu\xi + \frac{1}{2}\sigma^2\xi^2$$

our guess is verified for ξ a root of the equation above. There are two roots:

$$\xi_2 = -\frac{\mu + \sqrt{\mu^2 + 2\sigma^2\rho}}{\sigma^2} \qquad \xi_2 = \frac{-\mu + \sqrt{\mu^2 + 2\sigma^2\rho}}{\sigma^2},$$

where $\xi_1 < 0 < 1 < \xi_2$, this follows from $\rho > 0$ and the assumption $\rho - \left(\mu + \frac{1}{2}\sigma^2\right) > 0$. Joining we get the solution for our HJB equation:

$$V(x) = V^P(x) + A_1 e^{\xi_1 x} + A_2 e^{\xi_2 x}$$

Now we must determine the values of A_1 and A_2. To do so we first need to impose certain conditions on our value function.

(a) From optimality in exit it must be that: $V(x) \geq \Omega$.

(b) From feasibility it must be that: $V(x) \leq V^P(x) + \Omega$.

(c) Value matching implies: $V(x^\star) = \Omega$.

We will show that $A_2 = 0$. Suppose for a contradiction that $A_2 > 0$, then as $x \to \infty$ we have $e^{\xi_1 x} \to 0$ (because $\xi_1 < 0$), and $e^{\xi_2 x} \to \infty$ (because $\xi_2 > 0$), because $A_2 > 0$ this implies that V violates its upper bound. Now suppose for a contradiction that $A_2 < 0$, as before $e^{\xi_1 x} \to 0$ and $e^{\xi_2 x} \to \infty$, because $\xi_2 > 1$ the last term will grow faster than the first one, thus violating the lower bound (the value goes to $-\infty$). Then it must be that $A_2 = 0$.

Then we can obtain A_1 from the value matching condition:

$$\begin{aligned} V(x^\star) &= V^P(x^\star) + A_1 e^{\xi_1 x^\star} \\ \left(\Omega - V^P(x^\star)\right) e^{-\xi_1 x^\star} &= A_1 \end{aligned}$$

with this the solution is complete, given a value for $x^\star$. It is left to find such value, for that we make use of the smooth pasting condition:

$$\begin{aligned} V_x(x^\star) &= 0 \\ V_x^P(x^\star) + A_1 \xi_1 e^{\xi_1 x^\star} &= 0 \\ \frac{1}{\rho - \left(\mu + \frac{1}{2}\sigma^2\right)} + \left(\Omega - \frac{x^\star}{\rho - \left(\mu + \frac{1}{2}\sigma^2\right)}\right)\xi_1 &= 0 \end{aligned}$$

$$\frac{1}{\xi_1} + \left(\rho - \left(\mu + \frac{1}{2}\sigma^2\right)\right)\Omega = x^\star$$

21. The Kolmogorov Forward Equation

The last section of this part of the course develops the Kolmogorov Forward Equation, which describes the dynamics of the probability distribution of a random variable (given its initial value). Moreover, it characterizes the stationary distribution of the variable if such distribution exists. This is of particular importance for models with heterogenous agents because the distribution of the agents in the economy is obtained via the KFE.

Given some initial conditions x_0 and t_0 the objective is to characterize the probability distribution function $\varphi(x, t)$:

$$\Pr\left(x_t \in [a, b]\right) = \int_a^b \varphi(u, t)\, du$$

In order to characterize φ we first need to impose a process for x, and then use the random walk approximation. For simplicity:

$$dx = \mu dt + \sigma dW$$

In the random walk approximation the process varies in a period of length Δt by a magnitude of h, it increases with probability p or decreases with probability $1 - p$, where:

$$h = \sigma\sqrt{\Delta t} \qquad p = \frac{1}{2}\left(1 + \frac{\mu}{\sigma}\sqrt{\Delta t}\right)$$

From time $t - \Delta t$ to time t the process can reach a value x either by growing from $x - h$ or by decreasing from $x + h$. Then the probability (or more intuitively the fraction of the mass) at point x at time t is given by:

$$\varphi(x, t) = p\varphi(x - h, t - \Delta t) + (1 - p)\,\varphi(x + h, t - \Delta t)$$

We can approximate the elements of the right hand side with a second order Taylor expansion:

$$\varphi(x \pm h, t - \Delta t) \approx \varphi(x, t) - \Delta t \frac{\partial \varphi(x, t)}{\partial t} \pm h \frac{\partial \varphi(x, t)}{\partial x} + \frac{1}{2} h^2 \frac{\partial^2 \varphi(x, t)}{\partial x^2}$$

Terms of order higher than Δt are ignored. We can replace to get:

$$0 = -\Delta t \frac{\partial \varphi(x,t)}{\partial t} + (1-2p)\left(h\frac{\partial \varphi(x,t)}{\partial x}\right) + \frac{1}{2}h^2\frac{\partial^2 \varphi(x,t)}{\partial x^2}$$
$$0 = -\Delta t \frac{\partial \varphi(x,t)}{\partial t} - \frac{\mu}{\sigma}\sqrt{\Delta t}\left(\sigma\sqrt{\Delta t}\frac{\partial \varphi(x,t)}{\partial x}\right) + \frac{1}{2}\sigma^2\Delta t\frac{\partial^2 \varphi(x,t)}{\partial x^2}$$
$$0 = -\frac{\partial \varphi(x,t)}{\partial t} - \mu\frac{\partial \varphi(x,t)}{\partial x} + \frac{1}{2}\sigma^2\frac{\partial^2 \varphi(x,t)}{\partial x^2}$$

which gives the KFE:

$$\frac{\partial \varphi(x,t)}{\partial t} = -\mu\frac{\partial \varphi(x,t)}{\partial x} + \frac{1}{2}\sigma^2\frac{\partial^2 \varphi(x,t)}{\partial x^2}$$

If x follows a more general diffusion process we can change the argument above to get:

$$\frac{\partial \varphi(x,t)}{\partial t} = -\frac{\partial\left[\mu(x,t)\,\varphi(x,t)\right]}{\partial x} + \frac{1}{2}\frac{\partial^2\left[\sigma(x,t)^2\,\varphi(x,t)\right]}{\partial x^2}$$

The KFE is specially useful for finding the stationary distribution of the process. In this case the distribution does not depend on time so the KFE is:

$$0 = -\frac{\partial\left[\mu(x,t)\,\varphi(x)\right]}{\partial x} + \frac{1}{2}\frac{\partial^2\left[\sigma(x,t)^2\,\varphi(x)\right]}{\partial x^2}$$

This equation can be integrated once to get:

$$c_1 = -2\mu(x,t)\,\varphi(x) + \frac{\partial\left[\sigma(x,t)^2\,\varphi(x)\right]}{\partial x}$$

where c_1 is a constant of integration (to be determined later). Then we can use the integrating factor:

$$s(x) = e^{-\int^x \frac{2\mu(z,t)}{\sigma^2(z,t)}dz}$$

By multiplying both sides by the integrating factor we get:

$$s(x)\,c_1 = e^{-\int^x \frac{2\mu(z,t)}{\sigma^2(z,t)}dz}\left(-2\mu(x,t)\,\varphi(x) + \frac{\partial\left[\sigma(x,t)^2\,\varphi(x)\right]}{\partial x}\right)$$

The RHS can be rewritten noting that:

$$\begin{aligned}\frac{d}{dx}\left[s(x)\,\sigma^2(x,t)\,\varphi(x)\right] &= \frac{d}{dx}\left[e^{-\int^x \frac{2\mu(z,t)}{\sigma^2(z,t)}dz}\sigma^2(x,t)\,\varphi(x)\right] \\ &= \frac{d}{dx}\left[e^{-\int^x \frac{2\mu(z,t)}{\sigma^2(z,t)}dz}\right]\sigma^2(x,t)\,\varphi(x) + s(x)\,\frac{d}{dx}\left[\sigma^2(x,t)\,\varphi(x)\right] \\ &= -\frac{2\mu(x,t)}{\sigma^2(x,t)}\sigma^2(x,t)\,\varphi(x) + s(x)\,\frac{d}{dx}\left[\sigma^2(x,t)\,\varphi(x)\right] \\ &= -2\mu(x,t)\,\varphi(x) + s(x)\,\frac{d}{dx}\left[\sigma^2(x,t)\,\varphi(x)\right]\end{aligned}$$

Then we get:

$$s(x)\,c_1 = \frac{d}{dx}\left[s(x)\,\sigma^2(x,t)\,\varphi(x)\right]$$

Integrating again:

$$c_1\int^x s(y)\,dy + c_2 = s(x)\,\sigma^2(x,t)\,\varphi(x)$$

rearranging gives:

$$\varphi(x) = \frac{1}{s(x)\,\sigma^2(x,t)}\left(c_1\int^x s(y)\,dy + c_2\right)$$

where $\int^x f(\xi)\,d\xi = F(x)$, being F the antiderivative of f.

Example 21.1. Dynamics and Barriers Consider a brownian motion with two reflecting barriers $\overline{x}$ and $\underline{x}$. The process behaves as $dx = \mu dt + \sigma dW$ for $x \in (\underline{x}, \overline{x})$, but is kept in those bounds by force. In terms of the random walk representation that means that starting at $\overline{x} - h$ the process stays at $\overline{x} - h$ with probability p, instead of taking a step up, and goes down to $\overline{x} - 2h$ with probability $1 - p$. Similarly for $\overline{x} + h$.

The KFE applies for any point in the interior of the domain, so for $x \in (\underline{x}, \overline{x})$ we have:

$$\frac{\partial\varphi(x,t)}{\partial t} = -\mu\frac{\partial\varphi(x,t)}{\partial x} + \frac{1}{2}\sigma^2\frac{\partial^2\varphi(x,t)}{\partial x^2}$$

Moreover, because we are interested in the stationary behavior of the process we know that the distribution does not depend on time, which results in:

$$0 = -\mu\frac{\partial\varphi(x)}{\partial x} + \frac{1}{2}\sigma^2\frac{\partial^2\varphi(x)}{\partial x^2}$$

or better:

$$\varphi'(x) = \frac{1}{2}\frac{\sigma^2}{\mu}\varphi''(x)$$

We can solve this equation:

$$\varphi(x) = Ae^{\gamma x} + B$$

where $\gamma = \frac{2\mu}{\sigma^2}$ and A and B are constants to be determined. To find them we can make use of the boundary conditions implied by the barriers.

From the random walk approximation we can derive the following equation for the upper bound:

$$\begin{aligned}\varphi(\overline{x} - h) &= p\varphi(x - h) + p\varphi(x - 2h)\\ (1 - p)\varphi(\overline{x} - h) &= p\varphi(x - 2h)\end{aligned}$$

Using now a second order Taylor expansion around $\overline{x} - h$:

$$\begin{aligned}(1 - p)\varphi(\overline{x} - h) &= p\left(\varphi(\overline{x} - h) - h\varphi'(\overline{x} - h) + \frac{1}{2}h^2\varphi''(\overline{x} - h)\right)\\ (1 - 2p)\varphi(\overline{x} - h) &= -ph\varphi'(\overline{x} - h) + p\frac{1}{2}h^2\varphi''(\overline{x} - h)\\ -\frac{\mu}{\sigma}\sqrt{\Delta t}\varphi(\overline{x} - h) &= -\frac{1}{2}\left(1 + \frac{\mu}{\sigma}\sqrt{\Delta t}\right)\sigma\sqrt{\Delta t}\varphi'(\overline{x} - h) + \frac{1}{4}\left(1 + \frac{\mu}{\sigma}\sqrt{\Delta t}\right)\sigma^2\Delta t\varphi''(\overline{x} - h)\\ -\frac{2\mu}{\sigma^2}\varphi(\overline{x} - h) &= -\left(1 + \frac{\mu}{\sigma}\sqrt{\Delta t}\right)\varphi'(\overline{x} - h) + \frac{1}{2}\left(1 + \frac{\mu}{\sigma}\sqrt{\Delta t}\right)\sigma\sqrt{\Delta t}\varphi''(\overline{x} - h)\end{aligned}$$

taking $\Delta t \to 0$ we get:

$$\begin{aligned}\frac{2\mu}{\sigma^2}\varphi(\overline{x}) &= \varphi'(\overline{x})\\ \gamma\varphi(\overline{x}) &= \varphi'(\overline{x})\end{aligned}$$

Replacing for the solution of φ we find that $B = 0$. Then A is found to guarantee that φ integrates to one. This results in:

$$\varphi(\overline{x}) = \frac{\gamma e^{\gamma x}}{e^{\gamma\overline{x}} - e^{\gamma\underline{x}}}$$

22. Application: Real Options

Consider the problem of a firm that is thinking about investing in a new project. The payoff that the project generates varies stochastically, but its cost is fixed. To be precise: the firm can, at any point in time, pay a fixed cost I to invest on a project that will have a payoff $x(t)$. Firm's investment opportunity is a perpetual call option, that is, the right but not the obligation to buy a share of some asset at a pre-specified price.

The payoff is assumed to follow a geometric brownian motion, so that:

$$dx = \mu x dt + \sigma x dW$$

The firm discounts the future at a rate ρ, so the problem of the firm is:

$$V(x_0) = \max_T E\left[(x(T) - I)\, e^{-\rho T} | x(0) = x_0\right]$$

To fix ideas we can first solve for the deterministic case. For this we set $\sigma = 0$, which implies that $x(T) = x_0 e^{\mu t}$, for some initial value x_0. Then:

$$V(x_0) = \max_T \left(x_0 e^{\mu T} - I\right) e^{-\rho T}$$

The following results follow:

(a) If $\mu \leq 0$ then the payoff x is decreasing (or constant), so it is better to invest immediately if $x_0 > I$, or never to invest if $x_0 \leq I$. This implies that:

$$V(x_0) = \max\{x_0 - I, 0\}$$

(b) If $0 < \mu \leq \rho$ then x is growing, so the value of the firm (the value of holding the option to invest) is positive, even if initially $x_0 < I$. Eventually $x > I$.

(i) The optimal time is given by:

$$\begin{aligned} \frac{\partial \left(x_0 e^{\mu T} - I\right) e^{-\rho T}}{\partial T} &= 0 \\ -(\rho - \mu)\, x_0 e^{-(\rho-\mu)T} + \rho I e^{-\rho T} &= 0 \\ \frac{1}{\mu} \ln \frac{\rho I}{(\rho - \mu)\, x_0} &= T \end{aligned}$$

$$T = \max\left\{\frac{1}{\mu} \ln \frac{\rho I}{(\rho - \mu)\, x_0}, 0\right\}$$

(ii) In some cases it is best to invest immediately. This happens if:

$$\begin{aligned}\frac{1}{\mu}\ln\frac{\rho I}{(\rho-\mu)x_0} &\leq 0\\ \frac{\rho I}{(\rho-\mu)x_0} &\leq 1\\ x^\star = \frac{\rho I}{\rho-\mu} &\leq x_0\end{aligned}$$

Higher μ increases the threshold value of x_0. Thus inducing longer waits.

(iii) Joining we get:

$$V(x_0) = \begin{cases}\frac{\mu}{\rho-\mu}I\left(\frac{(\rho-\mu)x_0}{\rho I}\right)^{\frac{\rho}{\mu}} & \text{if } x_0 \leq \frac{\rho I}{\rho-\mu}\\ x_0 - I & \text{otw}\end{cases}$$

(c) If $\mu > \rho$ then the payoff grows faster than the firm discount of the future, which implies that the firm wants to wait forever.

Now we can solve the stochastic version of the problem. It is no longer possible to find $T^\star$ directly, but we can still find the threshold value $x^\star$. To do it we first define the HJB equation, recall from equation (20.9) that:

$$\rho V dt = E\left[dV\right]$$

(noting that the instantaneous payoff before investing is zero). We can use Ito's lemma to expand the RHS:

$$\rho V = \mu x V' + \frac{1}{2}\sigma^2 x^2 V''$$

There are three boundary conditions that must hold:

$$V(0) = 0 \qquad V\left(x^\star\right) = x^\star - I \qquad V'\left(x^\star\right) = 1$$

The first one follows from 0 being an absorbing state (because of the properties of the geometric brownian motion). The second one is value matching and the third one is smooth pasting.

We guess that the solution is of the form:

$$V(x) = Ax^\beta$$

for some A and β to be found later. This clearly solves the HJB equation. Replacing we can

solve for β:

$$\rho V = \mu x V' + \frac{1}{2}\sigma^2 x^2 V''$$
$$\rho A x^\beta = \mu\beta A x^\beta + \frac{1}{2}\sigma^2 A\beta\left(\beta - 1\right) x^\beta$$
$$\rho = \mu\beta + \frac{1}{2}\sigma^2\beta\left(\beta - 1\right)$$
$$0 = -\rho + \left(\mu - \frac{1}{2}\sigma^2\right)\beta + \frac{1}{2}\sigma^2\beta^2$$

β is then found from the roots of this equation:

$$\beta_1 = \frac{1}{2} - \frac{\mu}{\sigma^2} + \sqrt{\left(\frac{1}{2} - \frac{\mu}{\sigma^2}\right)^2 + 2\frac{\rho}{\sigma^2}} > 1$$
$$\beta_2 = \frac{1}{2} - \frac{\mu}{\sigma^2} - \sqrt{\left(\frac{1}{2} - \frac{\mu}{\sigma^2}\right)^2 + 2\frac{\rho}{\sigma^2}} < 0$$

There are two distinct roots so the solution to the HJB equation is in general:

$$V\left(x\right) = A_1 x^{\beta_1} + A_2 x^{\beta_2}$$

But in order for the first boundary condition to hold we need that $A_2 = 0$, because, with $\beta_2 < 0$, we could not evaluate the function otherwise. This leaves us with only one root, which we denote β, and one constant A that we find below.

Replacing on the value matching and smooth pasting conditions we get:

$$A\left(x^\star\right)^\beta = x^\star - I \qquad \beta A\left(x^\star\right)^{\beta-1} = 1$$

Solving for $A = \frac{\left(x^\star\right)^{1-\beta}}{\beta}$ and replacing on the first equation we get:

$$x^\star = \beta x^\star - \beta I$$
$$x^\star = \frac{\beta I}{\beta - 1}$$

which also gives the value of A.

In the optimal strategy the firm does not invest when $x^\star$ is equal to I (when the net present value of investing becomes positive), but instead there is wedge between the cost of investing and the value of investing. The wedge is given because the firm has to be compensated for giving up the option to wait and see if the value increases even further.

Comparative Statics [Optional]

The threshold value $x^\star$ depends on the parameters of the model through β. Although we have an explicit solution for β in this case, that is not always the case. Nevertheless we can use the quadratic equation that gives rise to β to run comparative statics.

Let Q be the quadratic equation, so that:

$$Q = -\rho + \left(\mu - \frac{1}{2}\sigma^2\right)\beta + \frac{1}{2}\sigma^2\beta^2$$

we want to know how β depends on σ. Taking total differentials we get:

$$\frac{dQ}{d\sigma} = \frac{\partial Q}{\partial \beta}\frac{d\beta}{d\sigma} + \frac{\partial Q}{\partial \sigma} = 0$$

where the derivatives are evaluated at the positive root β found above. This expression gives:

$$\frac{d\beta}{d\sigma} = -\frac{\left(\frac{\partial Q}{\partial \sigma}\right)}{\left(\frac{\partial Q}{\partial \beta}\right)}$$

Signing the numerator is easy, because $\frac{\partial Q}{\partial \sigma} = \sigma\beta\,(\beta - 1) > 0$, we know it is positive because the positive root β is higher than 1. Signing the denominator requires us to know the shape of Q. It can be easily shown that Q is increasing at β_1:

$$\frac{\partial Q}{\partial \beta} = \mu - \frac{1}{2}\sigma^2 + \sigma^2\beta = \sigma^2\sqrt{\left(\frac{1}{2} - \frac{\mu}{\sigma^2}\right)^2 + 2\frac{\rho}{\sigma^2}} > 0$$

Then:

$$\frac{d\beta}{d\sigma} < 0$$

This means that higher variance (more uncertainty over the payoff of investing) reduces β, which in turn increases $\frac{\beta}{\beta - 1}$. So the wedge between $x^\star$ and I increases with uncertainty, in other words the firm will need a larger return on the investment in order to invest.

23. Application: Menu cost Stokey (2009, Ch. 7)

Consider a firm whose profit flow at any date t depends on its relative price, that is: the ratio of its own nominal price to an aggregate (industry-wide or economy-wide) price index, where the latter is a geometric Brownian motion. Recall that if the price follows a GBM then its log follows a brownian motion. It is then convenient to work with the prices in log form. Let $p(t)$ be the log of the firm's nominal price and $\overline{p}(t)$ the log of the aggregate price index. Then:

$$d\overline{p} = -\mu dt + \sigma dW_p$$

The initial value for the firm's (log) nominal price p_0 is given. The firm can change its nominal price at any time, but to do so it must pay a fixed adjustment cost $c > 0$. This cost is constant over time and measured in real terms. Because control entails a fixed cost, the firm adjusts the price only occasionally and by discrete amounts.

The problem of the firm is to choose when to adjust the price, and by how much. One can see this as a problem of choosing the (random) times at which to adjust the price, or of choosing an inaction region, such that the price is adjusted when some condition is met.

The problem can be formulated in terms of that one state variable because the profit flow at any date depends only on the firm's relative price. Let:

$$z(t) = p(t) - \overline{p}(t)$$

When the firm adjusts its price the variable z jumps. Part of the problem will be to find the optimal value $z^\star$ to which z is set when the firm decides to take action. Between adjustments z evolves only with $\overline{p}$, so we have (for any time at which there is no adjustment):

$$dz = \mu dt + \sigma dW$$

where $dW = -dW_p$.

The profit flow of the firm, $\pi(z)$, is a stationary function of its relative price z, and profits are discounted at a constant interest rate r. The following restrictions on π, r, c and the parameters μ, σ^2 insure that the problem is well behaved:

(a) $r, c, \sigma^2 > 0$

(b) π is continuous everywhere, strictly increasing on $(-\infty, 0)$ and strictly decreasing on $(0, \infty)$.

    (i) The location of the peak of π at 0 is arbitrary.

We will assume that π takes the following form:

$$\pi(z) = \begin{cases} \pi_0 e^{\eta_+ z} & \text{if } z \geq 0 \\ \pi_0 e^{\eta_- z} & \text{if } z < 0 \end{cases}$$

where $\eta_+ < 0 < \eta_-$.

These assumptions imply that it is optimal to change the price if z gets too low or too high. Then the inaction region is: $(\underline{z}, \bar{z})$.

The HJB equation for $z \in (\underline{z}, \bar{z})$ is:

$$\rho V(z) = \pi(z) + \mu V'(z) + \frac{1}{2}\sigma^2 V''(z)$$

The boundary conditions for V are value matching and smooth pasting at $\underline{z}$ and $\bar{z}$:

$$\begin{aligned} V(\underline{z}) &= V\left(z^\star\right) - c \\ V(\bar{z}) &= V\left(z^\star\right) - c \\ V'(\underline{z}) &= 0 \\ V'(\bar{z}) &= 0 \end{aligned}$$

while $z^\star$ is optimally found to maximize V. So it must satisfy:

$$V'\left(z^\star\right) = 0$$

The solution to the HJB equation is, just as before:

$$V(z) = V^p(z) + A_1 e^{\xi_1 z} + A_2 e^{\xi_2 z}$$

where V^p is a particular solution and $H(z) = e^{\xi z}$ is a solution to the homogeneous equation.

Finding the particular solution is not trivial. Stokey proposes the following solution:

$$W(z) = E\left[\int_0^T e^{-rt}\pi(z(t))\,dt\right]$$

where T is the (random) time at which the price will be adjusted. W gives then the expected discounted value of the profits until the next adjustment. This function is difficult to deal with, because T is not a real number, but instead a random variable. It is however possible to exploit this to express W as an integral over values of z. This goes beyond what we are

covering, the details are available in Stokey (2009, Sec. 3.5). Critically it can be shown that:

$$W\left(\underline{z}\right) = W\left(\bar{z}\right) = 0$$

because there is no time until the next adjustment. It will occur in that instant.

This simplifies the value matching conditions to:

$$A_1 e^{\xi_1 \underline{z}} + A_2 e^{\xi_2 \underline{z}} = V\left(z^\star\right) - c$$
$$A_1 e^{\xi_1 \overline{z}} + A_2 e^{\xi_2 \overline{z}} = V\left(z^\star\right) - c$$

Unfortunately we cannot further solve this problem.

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